MA2001-7 Cálculo en Varias Variables
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Auxiliar 9: Repaso Materia

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P1. Sea $f : \mathbb{R}^2 \to \mathbb{R}^2$ una función de clase C^2 armónica.
Considere dos funciones $u, v : \mathbb{R}^2 \to \mathbb{R}$ de clase C^2, que cumplen la siguiente condición:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Demuestre que $g(x, y) = f(u(x, y), v(x, y))$ es armónica.

P2. Sea $g : \mathbb{R} \to \mathbb{R}^2$ una función diferenciable y $f : \mathbb{R}^2 \to \mathbb{R}^2$ dada por:

$$F(x, y) = (ax + by, cx - dy)$$

Con $a, b, c, d \in \mathbb{R}$ tales que, $ad + bc \neq 0$
Si además $g\cancel{(0,0) = 0}$ y se tiene que:

$g(0) = (0,0)$

$$D(f \ o \ g)(x) = \begin{bmatrix} \sin(x) \\ \cos(2x) \end{bmatrix}$$

Encuentre g de forma explicita en función de a, b, c y d.

P3. Sea $f : \mathbb{R}^2 \to \mathbb{R}$ la función definida por $f(x, y) = x^2 + \cos(x) + y^2 - y$

a) Encuentre y estudie la naturaleza de los puntos críticos de f.

b) Demuestre que f alcanza su mínimo global.
Indicación: Estudie la convexidad de f.

P4. Considere la función $f(x, y) = (1 - 2x^2 - y^2)^2$. Determine si existen puntos de mínimo o máximo globales de f en $\mathbb{R}^2$ y en tal caso encuéntrelos.

P1) $\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0$

PDA

$$\frac{\partial^2 g}{\partial x^2}(x,y) + \frac{\partial^2 g}{\partial y^2}(x,y) = 0$$

$$\frac{\partial g}{\partial x} = \frac{\partial f(u,v)}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f(u,v)}{\partial u} \cdot \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial f(u,v)}{\partial u} + \frac{\partial^2 f(u,v)}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 +$$

$$2\frac{\partial^2 f}{\partial u \partial v}(u,v)\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+$$

$$\frac{\partial^2 v}{\partial x^2}\frac{\partial f}{\partial v}(u,v)+\frac{\partial^2 f}{\partial v^2}(u,v)\left(\frac{\partial v}{\partial x}\right)^2$$

$$\frac{\partial^2 g}{\partial y^2}=\frac{\partial^2 u}{\partial y^2}\frac{\partial f}{\partial u}(u,v)+\frac{\partial^2 f}{\partial u^2}(u,v)\left(\frac{\partial u}{\partial y}\right)^2$$

$$+2\frac{\partial^2 f}{\partial u \partial v}(u,v)\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+$$

$$\frac{\partial^2 v}{\partial y^2}\cdot\frac{\partial f}{\partial v}(u,v)+\frac{\partial^2 f}{\partial v^2}(u,v)\left(\frac{\partial v}{\partial y}\right)^2$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= -\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 g}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}\frac{\partial f}{\partial u}(u,v) + \frac{\partial^2 f}{\partial u^2}(u,v)\left(\frac{\partial v}{\partial x}\right)^2$$

$$-2\frac{\partial^2 f}{\partial u \partial v}(u,v)\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} +$$

$$-\frac{\partial^2 v}{\partial x^2}\frac{\partial f}{\partial v}(u,v) + \frac{\partial^2 f}{\partial v^2}(u,v)\left(\frac{\partial u}{\partial x}\right)^2$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial f}{\partial u}(u,v) + \frac{\partial^2 f}{\partial u^2}(u,v)\left(\frac{\partial u}{\partial x}\right)^2 +$$

$$2\frac{\partial^2 f}{\partial u \partial v}(u,v)\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} +$$

$$\frac{\partial^2 v}{\partial x^2}\frac{\partial f}{\partial v}(u,v) + \frac{\partial^2 f}{\partial v^2}(u,v)\left(\frac{\partial v}{\partial x}\right)^2$$

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \underbrace{\left(\frac{\partial^2 f}{\partial v^2} + \frac{\partial^2 f}{\partial u^2}\right)}_{0}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 0$$

P2. Sea $g : \mathbb{R} \to \mathbb{R}^2$ una función diferenciable y $f : \mathbb{R}^2 \to \mathbb{R}^2$ dada por:

$$\cancel{F}\, f(x, y) = (ax + by, cx - dy)$$

Con $a, b, c, d \in \mathbb{R}$ tales que, $ad + bc \neq 0$
Si además $g\cancel{(0,0) = 0}$ y se tiene que:

$$g(0) = (0,0) \qquad D(f \ o \ g)(x) = \begin{bmatrix} \sin(x) \\ \cos(2x) \end{bmatrix}$$

Encuentre g de forma explicita en función de a, b, c y d.

$$D(f \circ g)(x) = Df(g(x)) \cdot g'(x)$$

$$Df(y) = \begin{bmatrix} a & b \\ c & -d \end{bmatrix}, \quad g'(x) = \begin{pmatrix} g_1'(x) \\ g_2'(x) \end{pmatrix}$$

$$D(f \circ g)(x) = \begin{bmatrix} a & b \\ c & -d \end{bmatrix} \begin{pmatrix} g_1'(x) \\ g_2'(x) \end{pmatrix} = \begin{pmatrix} \sin(x) \\ \cos(2x) \end{pmatrix}$$

$$a\, g_1'(x) + b\, g_2'(x) = \sin(x)$$

$$c\, g_1'(x) - d\, g_2'(x) = \cos(2x)$$

$$\begin{bmatrix} a & b \\ c & -d \end{bmatrix}^{-1} = \frac{-1}{ad+bc} \begin{bmatrix} -d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad+bc} \begin{bmatrix} d & b \\ c & -a \end{bmatrix}$$

$$\begin{pmatrix} g_1'(x) \\ g_2'(x) \end{pmatrix} = \frac{1}{ad+bc} \begin{bmatrix} d & b \\ c & -a \end{bmatrix} \begin{pmatrix} \sin(x) \\ \cos(2x) \end{pmatrix}$$

$$g_1'(x) = \frac{d \sin(x) + b \cos(2x)}{ad+bc}$$

Integrate $\int_0^x$

$$g_1(x) = \frac{-d \cos(x)\Big|_0^x + \frac{b}{2} \sin(2x)\Big|_0^x}{ad+bc}$$

$$g_1(x) = \frac{d\left(\cos(x) - 1\right) + \frac{b}{2}\sin(2x)}{ad + bc}$$

$$g_2'(x) = \frac{c\sin(x) - a\cos(2x)}{ad + bc}$$

$$\int_0^x \Rightarrow g_2(x) = \frac{c\left(\cos(x) - 1\right) - \frac{a}{2}\sin(2x)}{ad + bc}$$

P3) $f(x,y) = x^2 + \cos(x) + y^2 - y$

$$\nabla f(x,y) = \begin{pmatrix} 2x - \sin(x) \\ 2y - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad x = 0, \quad y = \frac{1}{2}$

$2x - \sin(x) = g(x)$ Se ~~ver~~ $\bar{x} = 0$ es un cero

$g'(x) = 2 - \cos(x) > 0 \Rightarrow g$ es est creciente

$\Rightarrow g$ es inyectiva

$\Rightarrow \quad \bar{x} = 0$ es el único cero de g

$$H_f(x,y) = \begin{bmatrix} 2-\cos(x) & 0 \\ 0 & 2 \end{bmatrix}$$

$2 - \cos(x) > 0$

$2 > 0$

$\Rightarrow H_f$ es def. positiva

$\Rightarrow$ Tiene mínimo es global

$(0, \frac{1}{2})$ Sería un mínimo global

$\boxed{P4}$ $f(x,y) = (1-2x^2-y^2)^2$

$$\cancel{\nabla f = \begin{pmatrix} 2(1-2x^2-y^2)(-4x) \\ 2(1-2x^2-y^2)(-2y) \end{pmatrix}} = \Big(\; \Big.$$

$$\boxed{f(x,y) \geq 0} \quad , \forall (x,y) \in \mathbb{R}^2$$

Si $x >>$ muy grande o

$y >>$ muy grande

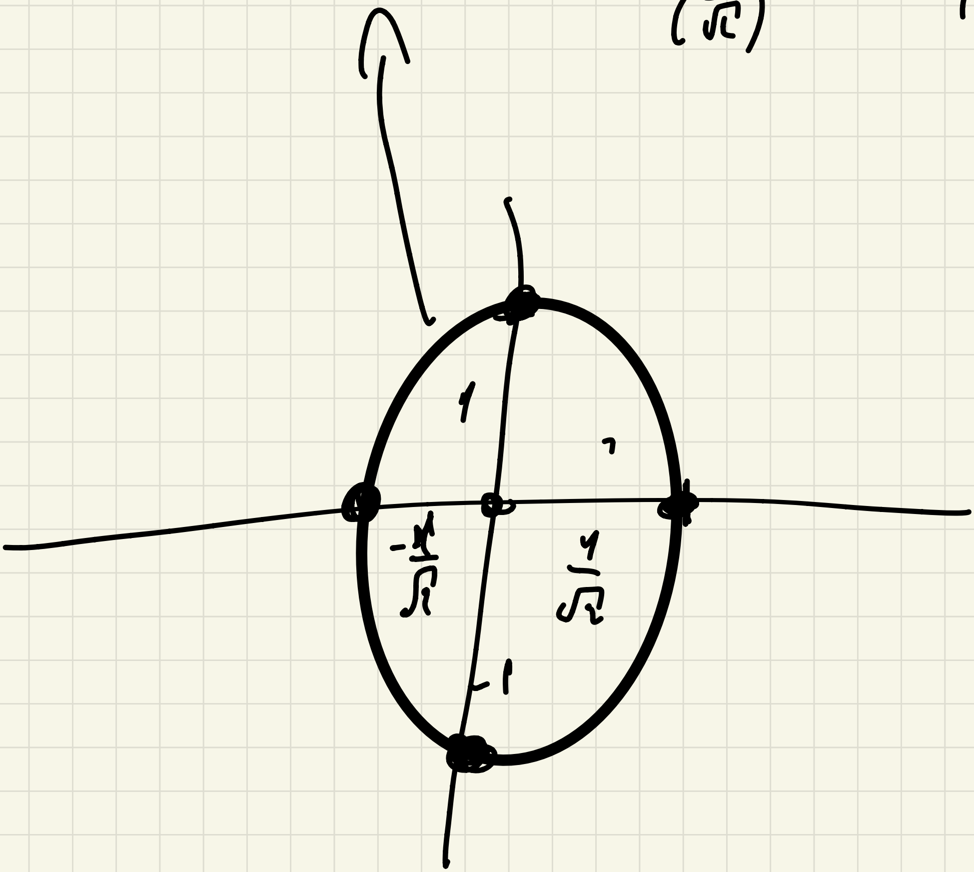
$$\lim_{\substack{x \to \infty \\ -\infty \\ y \to \infty \\ \|(x,y)\| \to \infty}} f(x,y) = \infty$$

No hay maximo global

$$(1 - 2x^2 - y^2)^2 = 0$$

$$\Leftrightarrow (1 - 2x^2 - y^2) = 0$$

$$1 = 2x^2 + y^2 = \frac{x^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{y^2}{1^2}$$



$f(x,y)$ tiene infinitos mínimos globales, que son la elipse $1 = 2x^2 + y^2$