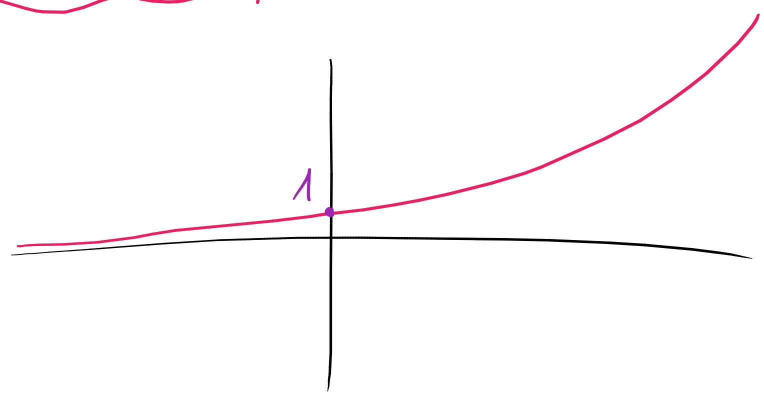
Función Exponencial

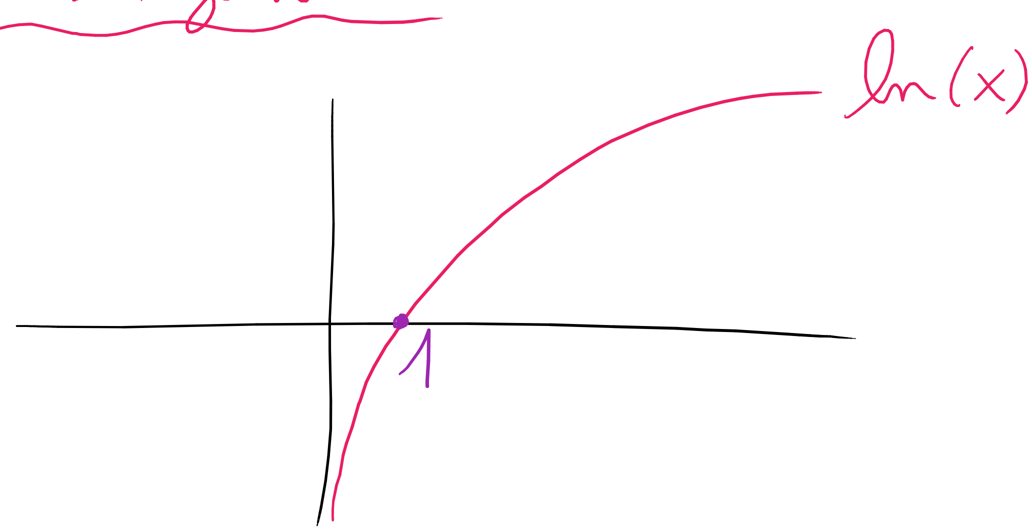
$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$



Desigualdades Fundamentales (útil para Sandwich):

- $\exp(x) \geq 1+x, \quad \forall x \in \mathbb{R}$
- $\exp(x) \leq \dfrac{1}{1-x}, \quad \forall x < 1$

Función Logaritmo



Propiedad muy útil:

- $\exp(\ln(x)) = x, \quad \forall x > 0$
- $\ln(\exp(x)) = x, \quad \forall x \in \mathbb{R}$

Desigualdades Fundamentales (útil para Sandwich)

$$1-\frac{1}{x} \leq \ln(x) \leq x-1, \quad \forall x > 0$$

P1 Calcule los siguientes límites de sucesiones:

a) $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=1}^{n} \exp\left(\frac{1}{n}\right)^{i}$

Sol: Sea $n \in \mathbb{N}$, $n > 1$ arbitrario. Notar que $0 < \frac{1}{n} < 1$, luego por las desigualdades fundamentales de $\exp(\cdot)$:

$$1 < 1+\frac{1}{n} \leq \exp\left(\frac{1}{n}\right) \leq \frac{1}{1-\frac{1}{n}}$$

$$\underset{\forall i \in \{1,\dots,n\}}{\Longrightarrow} \left(1+\frac{1}{n}\right)^{i} \leq \exp\left(\frac{1}{n}\right)^{i} \leq \left(\frac{1}{1-\frac{1}{n}}\right)^{i}$$

(pues $F(x) = x^{i}$ es creciente para $x > 1$)

$$\Longrightarrow \sum_{i=1}^{n}\left(1+\frac{1}{n}\right)^{i} \leq \sum_{i=1}^{n} \exp\left(\frac{1}{n}\right)^{i} \leq \sum_{i=1}^{n}\left(\frac{1}{1-\frac{1}{n}}\right)^{i}$$

$$\underset{\cdot\frac{1}{n}}{\Longrightarrow} \underbrace{\frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{1}{n}\right)^{i}}_{(A)} \leq \frac{1}{n}\sum_{i=1}^{n} \exp\left(\frac{1}{n}\right)^{i} \leq \underbrace{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{1-\frac{1}{n}}\right)^{i}}_{(B)} \qquad (*)$$

Notar que:

$$(A) = \frac{1}{n}\left(\sum_{i=0}^{n}\left(1+\frac{1}{n}\right)^{i} - \cancelto{1}{\left(1+\frac{1}{n}\right)^{0}}\right)$$

$$= \frac{1}{n}\sum_{i=0}^{n}\left(\frac{n+1}{n}\right)^{i} - \frac{1}{n}$$

$$\underset{(\Sigma \text{ geom})}{=} \frac{1}{n}\cdot\frac{\left(\frac{n+1}{n}\right)^{n+1} - 1}{\left(\frac{n+1}{n}\right) - 1} - \frac{1}{n}$$

$$= \cancel{\frac{1}{n}} \cdot \frac{\left(\frac{(n+1)^{n+1} - n^{n+1}}{n^{n+1}}\right)}{\left(\frac{\cancel{n}+1-\cancel{n}}{\cancel{n}}\right)} - \frac{1}{n}$$

$$= \frac{(n+1)^{n+1} - n^{n+1}}{n^{n+1}} - \frac{1}{n}$$

$$= \left(\frac{n+1}{n}\right)^{n+1} - 1 - \frac{1}{n}$$

$$= \underbrace{\left(1+\frac{1}{n}\right)^{n}}_{\to\,\exp(1)}\left(1+\underbrace{\frac{1}{n}}_{\to\,0}\right) - 1 - \underbrace{\frac{1}{n}}_{\to\,0}$$

$$\xrightarrow[n\to\infty]{} e - 1$$

Además:

$$B) = \frac{1}{n}\left(\sum_{i=0}^{n}\left(\frac{1}{1-\frac{1}{n}}\right)^{i} - \left(\frac{1}{1-\frac{1}{n}}\right)^{0}\right)$$

$$= \frac{1}{n}\sum_{i=0}^{n}\left(\frac{n}{n-1}\right)^{i} - \frac{1}{n}$$

$$\underset{\Sigma \text{ geom.}}{=} \frac{1}{n}\cdot\frac{\left(\frac{n}{n-1}\right)^{n+1} - 1}{\left(\frac{n}{n-1}\right) - 1} - \frac{1}{n}$$

$$= \frac{1}{n}\cdot\frac{\left(\frac{n^{n+1} - (n-1)^{n+1}}{(n-1)^{n+1}}\right)}{\left(\frac{\cancel{n}-\cancel{n}+1}{n-1}\right)} - \frac{1}{n}$$

$$= \frac{n-1}{n}\cdot\left(\left(\frac{n}{n-1}\right)^{n+1} - 1\right) - \frac{1}{n}$$

$$= \frac{n-1}{n}\cdot\left(\frac{1}{\left(\frac{n-1}{n}\right)^{n+1}} - 1\right) - \frac{1}{n}$$

$$= \frac{1}{\left(\frac{n-1}{n}\right)^{n}} - \frac{n-1}{n} - \frac{1}{n}$$

$$= \frac{1}{\underbrace{\left(1+\frac{(-1)}{n}\right)^{n}}_{\to\,\exp(-1)}} - 1 + \cancel{\frac{1}{n}} - \cancel{\frac{1}{n}}$$

$$\xrightarrow[n\to\infty]{} \frac{1}{e^{-1}} - 1 = \frac{1}{\frac{1}{e}} - 1 = e - 1.$$

$\therefore$ tomando $\lim\limits_{n\to\infty}$ en $(*)$ se concluye por Sandwich que:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \exp\left(\frac{1}{n}\right)^{i} = e - 1 \quad \blacksquare$$

b) $\lim\limits_{n\to\infty}\left(\dfrac{\ln(3n)}{\ln(n)}\right)^{\ln(n^2)}$

Sol:

$$\lim_{n\to\infty}\left(\frac{\ln(3n)}{\ln(n)}\right)^{\ln(n^2)}=\lim_{n\to\infty}\left(\frac{\ln(3)+\ln(n)}{\ln(n)}\right)^{2\ln(n)}$$

$$=\lim_{n\to\infty}\left(1+\frac{\ln(3)}{\ln(n)}\right)^{\ln(n)}\cdot\left(1+\frac{\ln(3)}{\ln(n)}\right)^{\ln(n)}$$

Cambio de variable:
$m=\ln(n)\underset{n\to\infty}{\longrightarrow}\infty$

$$=\lim_{m\to\infty}\underbrace{\left(1+\frac{\ln(3)}{m}\right)^{m}}_{\exp(\ln(3))=3}\cdot\underbrace{\left(1+\frac{\ln(3)}{m}\right)^{m}}_{\exp(\ln(3))=3}$$

$$=3\cdot 3=9$$

Límite de Funciones:

$$\lim_{x \to \bar{x}} f(x) = l \iff \forall (x_n)_{n\in\mathbb{N}} \subseteq Dom(f) : \; x_n \xrightarrow[n\to\infty]{} \bar{x} \Rightarrow f(x_n) \xrightarrow[n\to\infty]{} l$$

caracterización por sucesiones

$$\iff \forall \varepsilon > 0, \exists \delta > 0 : \; |x - \bar{x}| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

caracterización por $\varepsilon - \delta$

Importante:

Un límite existe $\iff$ límites laterales existen y son $=$

Continuidad:

f es continua en $\bar{x}$

$$\iff \lim_{x\to\bar{x}} f(x) = f(\bar{x})$$

(cercano a $\bar{x}$ puedo dibujar f sin levantar el lápiz del papel)

Límites Importantes:

- $\displaystyle \lim_{x\to 0} \frac{\operatorname{sen}(x)}{x} = 1$
- $\displaystyle \lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$
- $\displaystyle \lim_{x\to 1} \frac{\ln(x)}{x-1} = 1$
- $\displaystyle \lim_{x\to 0} \frac{e^x-1}{x} = 1$

P2 Calcule los siguientes límites de funciones.

a) $$\lim_{x\to 0} \frac{\sin(4x) - x}{1-\cos(x)+2x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \nearrow 1$$

$$= \lim_{x\to 0} \frac{\frac{\sin(4x)}{x} - 1}{\frac{1-\cos(x)}{x} + 2}$$

$$= \lim_{x\to 0} \frac{4 \cdot \frac{\sin(4x)}{4x} - 1}{x \cdot \frac{1-\cos(x)}{x^2} + 2} \quad (\#)$$

Notar que:

$$\lim_{x\to 0} \frac{\sin(4x)}{4x} \underset{\substack{\text{C.V.:} \\ y=4x \xrightarrow[x\to 0]{} 0}}{=} \lim_{y\to 0} \frac{\sin(y)}{y} = 1$$

límite conocido

Usando esto en $(\#)$ y álgebra de límites:

$$(\#) = \frac{4 \cdot 1 - 1}{0 \cdot \frac{1}{2} + 2} = \frac{4-1}{2} = \frac{3}{2}$$

b) $\lim_{x\to 1} x^{\frac{1}{1-x}}$

$= \lim_{x\to 1} \exp\left(\ln\left(x^{\frac{1}{1-x}}\right)\right)$

$= \lim_{x\to 1} \exp\left(\frac{1}{1-x}\ln(x)\right) \quad (\#)$

↓ propiedad $\ln(\cdot)$

Notar que:

límite conocido

$\lim_{x\to 1} \frac{1}{1-x}\ln(x) = -\lim_{x\to 1} \frac{\ln(x)}{x-1} = -1$

Usando esto en $(\#)$ y la continuidad de $\exp(\cdot)$ $(u\to\bar{u} \Rightarrow \exp(u)\to\exp(\bar{u}))$

$(\#) = \exp\left(\lim_{x\to 1} \frac{1}{1-x}\ln(x)\right)$

$= \exp(-1) = \frac{1}{e}$ ■