Einstein-Hilbert action

We start with the action of the form

$S_{EH} = \frac{1}{2\kappa}\int d^4x \sqrt{-g}\, R$, lets define $2\kappa = 1$ and begin to vary the action

$$\Rightarrow \delta S_{EH} = \int d^4x \left[\delta\sqrt{-g} \cdot R + \sqrt{-g}\,\delta R \right]$$

$$= \int d^4x \left[\frac{1}{2}\frac{-1}{\sqrt{-g}}\,\delta g\, R + \sqrt{-g}\,\delta(R_{\mu\nu} g^{\mu\nu}) \right]$$

$$= \int d^4x \left[-\frac{1}{2\sqrt{-g}} \cdot -g g^{\mu\nu}\delta g_{\mu\nu}\, R + \sqrt{-g}\,(\delta R_{\mu\nu}\cdot g^{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu}) \right]$$

$$= \int d^4x \left[-\frac{\sqrt{-g}}{2} g_{\mu\nu}\delta g^{\mu\nu} R + \sqrt{-g}\,\delta R_{\mu\nu}\cdot g^{\mu\nu} + \sqrt{-g}\,R_{\mu\nu}\delta g^{\mu\nu} \right]$$

$$= \int d^4x \left[-\frac{1}{2} g_{\mu\nu} R + R_{\mu\nu} \right] \sqrt{-g}\,\delta g^{\mu\nu} + \int d^4x \sqrt{-g}\,\delta R_{\mu\nu} g^{\mu\nu}$$

debo mostrar esto

Using the approximation at secon order $g_{\mu\nu} = \eta_{\mu\nu} + O(\delta x^2)$, where $\Gamma^{\alpha}_{\mu\nu} = 0$ but $\partial\Gamma$ is constant

≈ 0

Sabemos que $R_{ij} = \frac{\partial \Gamma^a_{ij}}{\partial x^a} - \frac{\partial \Gamma^a_{ai}}{\partial x^j} + \cancel{\Gamma^a_{ab}\Gamma^b_{ij}} - \cancel{\Gamma^a_{ib}\Gamma^b_{aj}}$

$$\Rightarrow \delta R_{\hat\mu\hat\nu} = \frac{\partial \delta\Gamma^{\hat\alpha}_{\hat\mu\hat\nu}}{\partial x^{\hat\alpha}} - \frac{\partial \delta\Gamma^{\hat\alpha}_{\hat\alpha\hat\mu}}{\partial x^{\hat\nu}} = \partial_{\hat\alpha}\delta\Gamma^{\hat\alpha}_{\hat\mu\hat\nu} - \partial_{\hat\nu}\delta\Gamma^{\hat\alpha}_{\hat\alpha\hat\mu}$$

Como $\Gamma \approx 0 \Rightarrow \partial_{\hat\alpha} \to \nabla_{\hat\alpha} \Rightarrow \delta R_{\hat\mu\hat\nu} = \nabla_{\hat\alpha}\delta\Gamma^{\hat\alpha}_{\hat\mu\hat\nu} - \nabla_{\hat\nu}\delta\Gamma^{\hat\alpha}_{\hat\alpha\hat\mu}$

$$\Rightarrow \delta R_{\mu\nu} = \nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu \delta\Gamma^\alpha_{\alpha\mu}$$

$$\Rightarrow \delta S_2 = \int d^4x \sqrt{-g}\,(\nabla_\alpha\delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu\delta\Gamma^\alpha_{\alpha\mu})\, g^{\mu\nu} = \int d^4x\sqrt{-g}\,(g^{\mu\nu}\nabla_\alpha\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\nu}\nabla_\nu\delta\Gamma^\alpha_{\alpha\mu})$$

$$= \int d^4x\sqrt{-g}\,(\nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\alpha_{\alpha\mu}))$$

$\nabla_\alpha g^{\mu\nu} = 0$

$$= \int d^4x\sqrt{-g}\,\nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu_{\nu\mu})$$

$$= \int d^4x\sqrt{-g}\,\nabla_\alpha U^\alpha$$

$$= \int d^4x\sqrt{-g}\left[\frac{\partial U^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\beta\alpha}U^\beta\right]$$

Computing Christoffel symbols

$$\triangleright\ \Gamma^{\alpha}_{\beta\alpha} = \frac{1}{2} g^{\alpha\gamma}\left(\partial_\alpha g_{\gamma\beta} + \partial_\beta g_{\gamma\alpha} + \partial_\gamma g_{\beta\alpha}\right)$$

wikipedia

$$\Gamma^{\alpha}_{\beta\alpha} = \partial_\beta \ln\sqrt{-g}$$

$$\Rightarrow \int d^4x \sqrt{-g}\left(\partial_\alpha U^\alpha + \partial_\beta(\ln\sqrt{-g})U^\beta\right) = \int d^4x \sqrt{-g}\left(\partial_\alpha U^\alpha + \frac{1}{\sqrt{-g}}\partial_\beta\sqrt{-g}\cdot U^\beta\right)$$

$$= \int d^4x\left(\sqrt{-g}\,\partial_\alpha U^\alpha + \partial_\alpha\sqrt{-g}\cdot U^\alpha\right)$$

$$= \int d^4x\, \partial_\alpha\left(\sqrt{-g}\,U^\alpha\right) = \oint_{\partial S} dS\, U^\alpha n_\alpha$$

And this term doesn't affect to the bulk, so we don't consider it.

$$\Rightarrow \int d^4x \left[-\frac{1}{2} g_{\mu\nu}R + R_{\mu\nu}\right]\sqrt{-g}\,\delta g^{\mu\nu} = 0$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

that are the Einstein field equations in the vaccum