MA2002-3 Cálculo Avanzado y Aplicaciones
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Auxiliar 12: Funciones Complejas, C-R y Series

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P1. Dada una función $f : \Omega \to \mathbb{C}$. Demuestre que si f y $\overline{f}$ son holomorfas, entonces f es una función constante.

P2. Sea $\Omega \subseteq \mathbb{C}$ un abierto no vació y $f : \Omega \to \mathbb{C}$, una función tal que:
-Existen $\hat{u}$ y $\hat{v} : \Omega \to \mathbb{R}$ funciones diferenciales en todo punto Ω que permiten expresar f en coordenadas cartesianas, $z = x + iy$ y $f(z) = \hat{u}(x, y) + i\hat{v}(x, y)$
-Existen u y $v : \Omega \to \mathbb{R}$ funciones diferenciales en todo punto Ω que permiten expresar f en coordenadas polares, $z = re^{i\theta}$, $r = \sqrt{x^2 + y^2}$ y $\theta = arctan(y/x)$ y $f(z) = u(r, \theta) + iv(r, \theta)$

a) Verifique que $u(r, \theta) = \hat{u}(r\cos(\theta), r\sin(\theta)))$ y $v(r, \theta) = \hat{v}(r\cos(\theta), r\sin(\theta)))$. Además pruebe que f es holomorfa en Ω si y solo si cumple:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad\qquad -\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta}$$

b) Con esto obtenga una formula para $f'(z)$ y compruebe que $f(z) = LOG(z) = \ln(r) + i\theta$, cumple que $f'(z) = \dfrac{1}{z}$

P3. Calcule la serie de potencias de $f(z) = \dfrac{1}{1-z}$, entorno a $z_0 = 0, -1, i$, también dibuje los discos de convergencia y relacione las distancias de los z_0 con $z = 1$.

P4. Calcule los radios de convergencia de las siguientes series:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{e^{in}}$

c) $\displaystyle\sum_{n=0}^{\infty} (n + 2^n) z^n$

d) $\displaystyle\sum_{n=0}^{\infty} (log(n))^2 z^n$

e) $\displaystyle\sum_{n=0}^{\infty} n^2 (z+2)^{2^n}$

f) $\displaystyle\sum_{n=0}^{\infty} (3 + (-1)^n)^n z^n$

Propuestos

1. Calcule la serie de potencias de $f(z) = \dfrac{2}{z^2+1}$ en torno a $z_0 = 1$.

Resumen

Condiciones de Cachy Riemann $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ y $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Teorema: Una función compleja es derivable si y solo si cumple las condiciones de Cauchy Riemann y es Frechet diferenciable .

Aproximación de Stirling: $n! \approx \sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n$

Radio de Convergencia $R = 1/\limsup_{n\to\infty} \sqrt[n]{|c_n|}$, con c_k el coeficiente de la serie centrada en z_0.
Obs: El radio puede depender de z_0 **Radio de convergencia**: Si este límite existe también se puede utilizar: $R = \limsup_{n\to\infty} \dfrac{|a_n|}{|a_{n+1|}}$

Diferenciable Diferenciable

$$f(z) = u(z) + i\,v(z) \;/\; \text{Si } z = x+iy$$

$$f(x,y) = u(x,y) + i\,v(x,y)$$

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \quad \text{existe}$$

Parte Real

Parte Compleja

$$\lim_{x,y\to \bar{x},\bar{y}} \frac{\|f(x,y)-f(\bar{x},\bar{y})\|}{\|(x,y)-(\bar{x},\bar{y})\|} \quad \text{existe}$$

P1. Dada una función $f : \Omega \to \mathbb{C}$. Demuestre que si f y $\overline{f}$ son holomorfas, entonces f es una función constante.

$f(z) = z$ sabemos que es holomorfa

$\overline{f(z)} = \overline{z}$ ¿puede ser holomorfa?

$\downarrow$

$\overline{f(x,y)} = x - iy$

$f(x,y) = u(x,y) + i\,v(x,y)$

Como f Holomorfa

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad y \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Como $\overline{f}$ Holomorfa $\overline{f}(x,y) = u(x,y) + i(-v(x,y))$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad y \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$\square$ y $\blacksquare \Rightarrow \dfrac{\partial v}{\partial y} = 0 = \dfrac{\partial u}{\partial x}$

No puede ser igual a algo y menos Algo salvo 0

$\bigcirc$ y $\bullet \Rightarrow \dfrac{\partial v}{\partial x} = 0 = \dfrac{\partial u}{\partial y}$

Como Todas las derivadas son 0

f es constante

P2. Sea $\Omega \subseteq \mathbb{C}$ un abierto no vació y $f : \Omega \to \mathbb{C}$, una función tal que:

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-Existen u y $v : \Omega \to \mathbb{R}$ funciones diferenciales en todo punto Ω que permiten expresar f en coordenadas polares, $z = re^{i\theta}$, $r = \sqrt{x^2 + y^2}$ y $\theta = arctan(y/x)$ y $f(z) = u(r, \theta) + iv(r, \theta)$

$\checkmark$

a) Verifique que $u(r, \theta) = \hat{u}(r\cos(\theta), r\sin(\theta)))$ y $v(r, \theta) = \hat{v}(r\cos(\theta), r\sin(\theta)))$. Además pruebe que f es holomorfa en Ω si y solo si cumple:

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b) Con esto obtenga una formula para $f'(z)$ y compruebe que $f(z) = LOG(z) = \ln(r) + i\theta$, cumple que $f'(z) = \frac{1}{z}$

a)

$r = \sqrt{x^2+y^2}$ Coordenadas polares

$\Rightarrow r\cos\theta = x$

$\theta = \text{Artan}(\theta)$ $\quad r\sin\theta = y$

$\hat{u}(r\cos\theta, r\sin\theta) = \hat{u}(x,y)$

$\hat{v}(r\cos\theta, r\sin\theta) = \hat{v}(x,y)$

$\hat{u}(x,y) = Re(f) = u(r,\theta)$

$\hat{v}(x,y) = Im(f) = v(r,\theta)$

Como u, v, $\hat{u}, \hat{v}$ son todas diferenciable (Asumiendo esto por enunciado)

f es Holomorfa $\Leftrightarrow$ Si cumple C-R

Por lo que probaremos

$$\frac{\partial \hat{u}}{\partial x} = \frac{\partial \hat{v}}{\partial y} \quad \text{y} \quad \frac{\partial \hat{u}}{\partial y} = -\frac{\partial \hat{v}}{\partial x} \iff$$

si cumple:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad -\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\hat{u}(x, y) = u(r, \theta)$$

Regla de la cadena

$$\frac{\partial \hat{u}}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r} \cdot \frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta} \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right)$$

$$= \frac{\partial u}{\partial r} \cdot \frac{r \cos\theta}{r} + \frac{\partial u}{\partial \theta} \frac{1}{x^2+y^2} (-y)$$

$$= \frac{\partial u}{\partial r} \cdot \cos\theta - \frac{\partial u}{\partial \theta} \frac{\sin\theta}{r}$$

$$\frac{\partial \hat{u}}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r} \cdot \frac{y}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta} \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) \left(\frac{x}{x}\right)$$

$$= \frac{\partial u}{\partial r} \sin\theta + \frac{\partial u}{\partial \theta} \cdot \frac{\cos\theta}{r}$$

$$\frac{\partial \hat{v}}{\partial x} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$= \frac{\partial v}{\partial r} \cdot \cos\theta - \frac{\partial v}{\partial \theta} \cdot \frac{\sin\theta}{r}$$

$$\frac{\partial \hat{v}}{\partial y} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$= \frac{\partial v}{\partial r} \cdot \sin\theta + \frac{\partial v}{\partial \theta} \cdot \frac{\cos\theta}{r}$$

Resumen

$$\frac{\partial \hat{u}}{\partial x} = \frac{\partial u}{\partial r} \cdot \cos\theta - \frac{\partial u}{\partial \theta} \frac{\sin\theta}{r}$$

$$\frac{\partial \hat{u}}{\partial y} = \frac{\partial u}{\partial r} \sin\theta + \frac{\partial u}{\partial \theta} \cdot \frac{\cos\theta}{r}$$

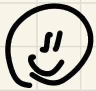
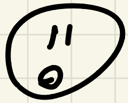
$$\frac{\partial \hat{v}}{\partial x} = \frac{\partial v}{\partial r} \cos\theta - \frac{\partial v}{\partial \theta} \cdot \frac{\sin\theta}{r}$$

$$\frac{\partial \hat{v}}{\partial y} = \frac{\partial v}{\partial r} \cdot \sin\theta + \frac{\partial v}{\partial \theta} \cdot \frac{\cos\theta}{r}$$

C-R

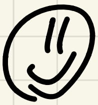
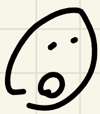
$$\frac{\partial \hat{u}}{\partial x} = \frac{\partial \hat{v}}{\partial y} \iff \underbrace{\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial \theta} \cdot \frac{1}{r}\right) \cos\theta = \left(\frac{\partial v}{\partial r} + \frac{\partial u}{\partial \theta} \cdot \frac{1}{r}\right) \sin\theta}_{!!}$$

$$\frac{\partial \hat{u}}{\partial y} = -\frac{\partial \hat{v}}{\partial x} \iff \underbrace{\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial \theta} \cdot \frac{1}{r}\right) \sin\theta = -\left(\frac{\partial v}{\partial r} + \frac{\partial u}{\partial \theta} \cdot \frac{1}{r}\right) \cos\theta}_{!!}$$

C-R $\iff$  y 

Idea: Nos gustaría que $\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial \theta}\cdot\frac{1}{r}\right) = 0$

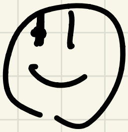
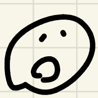
y $\left(\frac{\partial v}{\partial r} + \frac{\partial u}{\partial \theta}\cdot\frac{1}{r}\right) = 0$

 $\cdot \cos\theta \quad + $  $\cdot \sin\theta$

$$\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial \theta}\cdot\frac{1}{r}\right)\left(\cos\theta^2 + \sin\theta^2\right) = 0$$

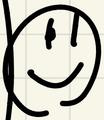
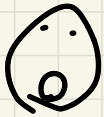
$$\Rightarrow \frac{\partial u}{\partial r} - \frac{\partial v}{\partial \theta}\cdot\frac{1}{r} = 0$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}\cdot\frac{1}{r}$$

 $\cdot (-\sin\theta) \quad + $  $\cdot \cos\theta$

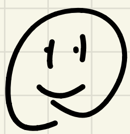
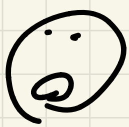
$$0 = \left(\frac{\partial v}{\partial r} - \left(-\frac{\partial u}{\partial \theta}\cdot\frac{1}{r}\right)\right)(-1)$$

$$\Rightarrow \frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}\cdot\frac{1}{r}$$

 y  $\Rightarrow \quad \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}\cdot\frac{1}{r}$

y $\quad \frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}\cdot\frac{1}{r}$

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}\cdot\frac{1}{r} \quad y \quad \frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}\cdot\frac{1}{r}$$

$\Rightarrow$  y 

Si pues queda $0 = 0$ al reemplazar los $\frac{\partial u}{\partial r}$ y $\frac{\partial v}{\partial r}$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial v}{\partial \theta} \cdot \frac{1}{r} \\ \frac{\partial v}{\partial r} &= -\frac{\partial u}{\partial \theta} \cdot \frac{1}{r} \end{aligned}} \iff \quad \iff \text{C-R}$$

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2) $$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad (1)$$

$$\cancel{= i \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}} \quad ?$$

$z = x + iy$

$$= \frac{\partial u}{i\,\partial y} + \frac{\cancel{i}}{\cancel{i}} \frac{\partial v}{\partial y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$f'(r,\theta) \overset{(1)}{=} \frac{\partial u}{\partial r}\cdot \cos\theta - \frac{\partial u}{\partial \theta}\cdot \frac{1}{r}\sin\theta$$

$$+\, i\left(\frac{\partial v}{\partial r}\cdot \cos\theta - \frac{\partial v}{\partial \theta}\cdot \frac{1}{r}\sin\theta\right)$$

$$\ln(z) = \underbrace{\ln(r)}_{u(r,\theta)} + \underbrace{i\theta}_{v(r,\theta)}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r} \quad , \quad \frac{\partial u}{\partial \theta} = 0$$

$$\frac{\partial v}{\partial r} = 0 \quad , \quad \frac{\partial v}{\partial \theta} = 1$$

f cumple C-R versión polar

Si: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ ✓ , $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ ✓

$$f'(r,\theta) = \frac{1}{r}\cos\theta - i\,\frac{1}{r}\sin\theta$$

$$= \frac{1}{r}\left(\cos\theta - i\sin\theta\right)$$

$$\cos(-\theta) + i\sin(-\theta)$$

$$= \frac{1}{r}e^{-i\theta}$$

$$= \frac{1}{re^{i\theta}} = \boxed{\frac{1}{z}}$$

$$\left(\ln(z)\right)' = \frac{1}{z}$$

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