MA2002-3 Cálculo Avanzado y Aplicaciones
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Auxiliar 9: Transformadas de Fourier

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P1. Sea $0 < \delta < \infty$. Considere la función$1_{[-\delta,\delta]}(x) = \begin{cases} 1 & \text{si } |x| \leq \delta \\ 0 & \text{si } |x| > \delta \end{cases}$

a) Demuestre que esta función es integrable y, luego, pruebe que $F(1_{[-\delta,\delta]})(s) = \sqrt{\frac{2}{\pi}} \frac{sin(\delta s)}{s}$

b) Se define la función $\Lambda(x)$ como $\Lambda(x) = \begin{cases} 0 & \text{si } |x| > 1 \\ 1 - |x| & \text{si } |x| \leq 1 \end{cases}$

Pruebe que

$$\left(1_{[-\frac{1}{2},\frac{1}{2}]} * 1_{[-\frac{1}{2},\frac{1}{2}]}\right)(x) = \Lambda(x)$$

c) Concluya que $F(\Lambda)(s) = \sqrt{\frac{8}{\pi}} \frac{sin^2(\frac{s}{2})}{s^2}$

P2. Sea $f(x) = \begin{cases} 1 & \text{si } |x| \leq 1 \\ 0 & \text{si } |x| > 1 \end{cases}$

a) Calcule la transformada de Fourier de f

b) Encuentre una función g tal que $\hat{g}(\xi) = \hat{f}(\xi)e^{-\xi^2}$, puede dejarla expresada como una integral .

P3. *a*) Sea $f(x) = e^{-ax}1_{x\geq 0}$, pruebe que $\hat{f}(x)(s) = \frac{1}{\sqrt{2\pi}} \frac{1}{a + is}$

b) Ahora calcule la transformada de $g(x) = \begin{cases} e^{-2x} & \text{si } x \geq 0 \\ e^{5x} & \text{si } x < 0 \end{cases}$

P4. Sean $a, b > 0$ considere la EDO

$$y'' + y = e^{-ax}1_{x\geq b}$$

Encuentre la solución en su forma integral sin transformadas.

Resumen

Recuerden que la transformada es lineal

1. $\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} f(x)dx$
2. $\check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} g(s)ds$
3. $\widehat{f'}(s) = is\widehat{f}(s)$
4. $\widehat{f^{(k)}}(s) = (is)^k \widehat{f}(s)$
5. $g'(x) = -ix\check{g}(x)$
6. $g^{(k)}(x) = (-ix)^k \check{g}(x)$
7. $f * g(x) = \int_{-\infty}^{\infty} f(s)g(x-s)ds$
8. $\widehat{f * g}(s) = \sqrt{2\pi}\hat{f}(s)\hat{g}(s)$
9. $f\widehat{(x - x_0)}(s) = e^{-isx_0}\hat{f}(s)$
10. $e^{is_0x}\widehat{f(x)}(s) = \hat{f}(s - s_0)$
11. $\widehat{f(ax)}(s) = \frac{1}{|a|}\hat{f}\left(\frac{s}{a}\right)$
12. $\widehat{f(-x)}(s) = \hat{f}(-s)$

$$\int_0^\infty e^{-isx}\,dx$$

$$= -\frac{e^{-isx}}{is}\Big|_0^\infty$$

$$= \frac{1}{is}$$

	$f(x)$	$\widehat{f}(s)$
1	$\begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\frac{1}{\sqrt{2\pi}} \frac{1}{1 + is}$
2	$e^{-ax^2}, \ a > 0$	$\frac{1}{\sqrt{2a}} e^{-\frac{s^2}{4a}}$
3	$e^{-a\|x\|}, \ a > 0$	$\sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}$
4	$\frac{1}{a^2 + x^2}, \ a > 0$	$\sqrt{\frac{2}{\pi}} \frac{1}{a} e^{-a\|s\|}$
5	$\begin{cases} k & \|x\| \leq b \\ 0 & \|x\| > b \end{cases}$	$\sqrt{\frac{2}{\pi}} k \frac{\operatorname{sen}(bs)}{s}$

$$e^{-i\lambda x} = \cos(-\lambda x) + i\sin(-\lambda x)$$

$$= \cos(\lambda x) - i\sin(\lambda x)$$

P1)

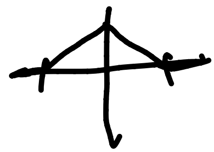
. Sea $0 < \delta < \infty$. Considere la función$1_{[-\delta,\delta]}(x) = \begin{cases} 1 & \text{si } |x| \leq \delta \\ 0 & \text{si } |x| > \delta \end{cases}$ indicatriz

a) Demuestre que esta función es integrable y, luego, pruebe que $F(1_{[-\delta,\delta]})(s) = \sqrt{\frac{2}{\pi}}\frac{sin(\delta s)}{s}$

b) Se define la función $\Lambda(x)$ como $\Lambda(x) = \begin{cases} 0 & \text{si } |x| > 1 \\ 1 - |x| & \text{si } |x| \leq 1 \end{cases}$

Pruebe que

$$\left(1_{[-\frac{1}{2},\frac{1}{2}]} * 1_{[-\frac{1}{2},\frac{1}{2}]}\right)(x) = \Lambda(x)$$



c) Concluya que $F(\Lambda)(s) = \sqrt{\frac{8}{\pi}}\frac{sin^2(\frac{s}{2})}{s^2}$

a) Porque $\mathbb{1}_{[-\delta,\delta]}(x)$ es integrable

Continuas Acotadas lo son ✓

$\mathbb{1}_{[-\delta,\delta]}(x)$ es Continua por Tramos

(Tiene una cantidad finita de discontinuidades)

$$F(f(x))(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ixs}\, dx$$

$$F\left(\mathbb{1}_{[-\delta,\delta]}(x)\right)(\nu) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathbb{1}_{[-\delta,\delta]}(x)\, e^{-ix\nu}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\delta}^{\delta} 1\cdot e^{-ix\nu}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\delta}^{\delta} \underbrace{\cos(x\nu)}_{(\text{Par})} - i\, \underbrace{\cancel{\sin(x\nu)}}_{(\text{Impar})\ 0}\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\delta} \cos(x\nu)\, dx$$

$$\sqrt{\frac{2}{\pi}}\, \frac{\sin(x\nu)}{\nu}\Bigg|_{0}^{\delta}$$

$$\boxed{\sqrt{\frac{2}{\pi}}\, \frac{\sin(\delta\nu)}{\nu}}$$

b) $\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} * \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} = \Lambda(x)$

$$= \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & \sim \end{cases}$$

$$f * g(x) = \int_{-\infty}^{\infty} f(y) \cdot g(x-y)\, dy$$

$$\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} * \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(x)$$

$$= \int_{-\infty}^{\infty} \underline{\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(y)} \cdot \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(x-y)\, dy =$$

$$\equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(x-y)\, dy$$

$z = x - y$

$\underline{dz = -dy}$

$$\int_{x+\frac{1}{2}}^{x-\frac{1}{2}} \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2}]}(z)(-dz)$$

$$\underline{g(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2}]}(z)\, dz}$$

$\underline{S: \; |x| > 1}$

$\underline{x > 1} \Rightarrow \left.\begin{array}{l} x + \frac{1}{2} > \frac{3}{2} \\ x - \frac{1}{2} > \frac{1}{2} \end{array}\right\} \Rightarrow z \in ?$

$$\Rightarrow \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2}]}(z)\, dz$$

$$\frac{1}{2} < \boxed{x - \frac{1}{2} < z < x + \frac{1}{2}}$$

$$\Rightarrow z \notin \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$\Rightarrow \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} 0 \, dz = 0$$

$$\underline{x < -1}$$

$$x - \frac{1}{2} < -\frac{3}{2} \quad y \quad x + \frac{1}{2} < -\frac{1}{2}$$

$$x - \frac{1}{2} < z < x + \frac{1}{2} < -\frac{1}{2}$$

$$\Rightarrow z \notin \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$\Rightarrow \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} 0 \, dz = 0$$

$$\Rightarrow |x| > 1 \qquad g(x) = 0$$

Caso $\quad 0 \leq x \leq 1$

$$-\frac{1}{2} \leq x - \frac{1}{2} \leq \frac{1}{2}$$

$$\frac{1}{2} \leq x + \frac{1}{2} \leq \frac{3}{2}.$$

$$x - \frac{1}{2} \leq z \leq x + \frac{1}{2}$$

$$g(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2}]}(z) \, dz$$

$$g(x) = \int_{x-\frac{1}{2}}^{\frac{1}{2}} \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2}]}(z)\,dz$$

¿ $z \in [-\frac{1}{2}, \frac{1}{2}]$? ✓

$$= \int_{x-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot dz$$

$$= x \Big|_{x-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{2} - x + \frac{1}{2}$$

$$= 1 - x = 1 - |x|$$

Caso $\quad -1 \leq x < 0$

$$-\frac{3}{2} \leq x - \frac{1}{2} < -\frac{1}{2}$$

$$-\frac{1}{2} \leq x + \frac{1}{2} < \frac{1}{2}$$

$$g(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2}]}(z)\,dz$$

$$= \int_{-\frac{1}{2}}^{x+\frac{1}{2}} \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2}]}(z)\,dz$$

$$= \int_{-\frac{1}{2}}^{x+\frac{1}{2}} dz = 1 + x$$

$$= 1 - |x|$$

Finalmente $g(x) = \Lambda(x)$

. Sea $0 < \delta < \infty$. Considere la función$1_{[-\delta,\delta]}(x) = \begin{cases} 1 & \text{si } |x| \leq \delta \\ 0 & \text{si } |x| > \delta \end{cases}$

a) Demuestre que esta función es integrable y, luego, pruebe que $F(1_{[-\delta,\delta]})(s) = \sqrt{\frac{2}{\pi}} \frac{sin(\delta s}{s}$

b) Se define la función $\Lambda(x)$ como $\Lambda(x) = \begin{cases} 0 & \text{si } |x| > 1 \\ 1 - |x| & \text{si } |x| \leq 1 \end{cases}$

Pruebe que

$$\left(1_{[-\frac{1}{2},\frac{1}{2}]} * 1_{[-\frac{1}{2},\frac{1}{2}]}\right)(x) = \Lambda(x)$$

c) Concluya que $F(\Lambda)(s) = \sqrt{\frac{8}{\pi}} \frac{sin^2(\frac{s}{2})}{s^2}$

c) $F(\Lambda)(s) = F\left(1_{[-\frac{1}{2},\frac{1}{2}]} * 1_{[-\frac{1}{2},\frac{1}{2}]}\right)(s)$

$$\overset{\text{Prop}}{=} \sqrt{2\pi}\left(F\left(1_{[-\frac{1}{2},\frac{1}{2}]}\right)(s)\right)^2$$

$$= \sqrt{2\pi}\left(\sqrt{\frac{2}{\pi}}\,\frac{\sin\left(\frac{s}{2}\right)}{s}\right)^2$$

$$= \sqrt{\frac{8}{\pi}}\,\frac{\sin^2\left(\frac{s}{2}\right)}{s^2}$$

P2. Sea $f(x) = \begin{cases} 1 & \text{si } |x| \leq 1 \\ 0 & \text{si } |x| > 1 \end{cases}$

a) Calcule la transformada de Fourier de f

b) Encuentre una función g tal que $\hat{g}(\xi) = \hat{f}(\xi)e^{-\xi^2}$, puede dejarla expresada como una integral .

a) Usando lo anterior con $\underline{\delta = 1}$

$$\hat{f}(\nu) = \sqrt{\frac{2}{\pi}} \frac{\sin(\nu)}{\nu}$$

b) $$\hat{g}(\xi) = \hat{f}(\xi) e^{-\xi^2}$$

$$g(x) = \hat{f}(\xi) \cdot e^{-\xi^2} \quad \hat{h}(\xi)$$

$$= \hat{h}(\xi)$$

$$g(x) = h(x)$$

$$\boxed{\tilde{T}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2}}$$ / Si lo tuviamos

$$\sqrt{2\pi} \cdot \hat{T}(\xi) \cdot \hat{f}(\xi) = \hat{f}(\xi) e^{-\xi^2}$$

Prop $\widehat{(T * f)(x)} = \widehat{g(x)}$

Ant: Transformada

$$\Rightarrow \quad g(x) = (T * f)(x)$$

2	$e^{-ax^2}, \ a > 0$	$\frac{1}{\sqrt{2a}} e^{-\frac{s^2}{4a}}$

$$\widehat{e^{-ax^2}} = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}$$

$$\widehat{\frac{e^{-\frac{x^2}{4}}}{\sqrt{2}}} = \sqrt{2}\, e^{-\xi^2} \Big/ \frac{1}{\sqrt{2\pi}}$$

$$\widehat{\frac{e^{-\frac{x^2}{4}}}{4\sqrt{\pi}}} = \frac{e^{-\xi^2}}{\sqrt{2\pi}} = \hat{f}(\xi)$$

Antitrasformando

$$\Rightarrow \boxed{T(x) = \frac{e^{-\frac{x^2}{4}}}{4\sqrt{\pi}}}$$

$$\widehat{(T * f)(x)}(\xi) = \hat{g}^{(x)}(\xi)$$

Per Antitrasformata

$$\Rightarrow (T * f)(x) = g^{(x)}$$

$$g(x) = \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{4}}}{4\sqrt{\pi}} \cdot \mathbb{1}_{[-1,1]}(x-y)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4}}}{4\sqrt{\pi}} \mathbb{1}_{[-1,1]}(y)\,dy$$

$$\boxed{g(x) = \int_{-1}^{1} \frac{e^{-\frac{(x-y)^2}{4}}}{4\sqrt{\pi}}\,dy}$$

P3. a) Sea $f(x) = e^{-ax}1_{x\geq 0}$, pruebe que $\hat{f}(x)(s) = \frac{1}{\sqrt{2\pi}}\frac{1}{a+is}$

$a>0$

b) Ahora calcule la transformada de $g(x) = \begin{cases} e^{-2x} & \text{si } x\geq 0 \\ e^{5x} & \text{si } x<0 \end{cases}$

$g(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\frac{1}{\sqrt{2\pi}}\frac{1}{1+is}$

$+\ \widehat{f(ax)}(s) = \frac{1}{\vert a\vert}\hat{f}\left(\frac{s}{a}\right)$

$$g(ax) = \begin{cases} e^{-ax} & \cancel{a}x \geq 0 \\ 0 & \cancel{a}x < 0 \end{cases}$$

$$= e^{-ax}\mathbb{1}_{x\geq 0}$$

$$\hat{f}(x)(\nu) = \widehat{g(ax)}(\nu) = \frac{1}{a}\hat{g}(x)\left(\frac{\nu}{a}\right)$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\frac{1}{1+i\left(\frac{\nu}{a}\right)}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{a+i\nu}$$

$$h(x) = e^{-2x} \mathbb{1}_{x \geq 0}$$

$$\hat{h}(x)(\nu) = \frac{1}{\sqrt{2\pi}} \frac{1}{2+i\nu}$$

$$l(x) = e^{5x} \mathbb{1}_{x<0}$$

$$p(x) = l(-x) = e^{-5x} \mathbb{1}_{x>0}$$

$$\Rightarrow \quad \hat{p}(x)(\nu) = \frac{1}{\sqrt{2\pi}} \frac{1}{5+i\nu}$$

$$l(x) = p(-x)$$

$$\widehat{l(x)}(\nu) = \widehat{p(-x)}(\nu) = \widehat{p(x)}(-\nu)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{5-i\nu}$$

$$g(x) = l(x) + h(x)$$

Linealidad

$$\widehat{g(x)}(\nu) = \widehat{l(x)}(\nu) + \widehat{h(x)}(\nu)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2+i\nu} + \frac{1}{5-i\nu}\right)$$

P4. Sean $a, b > 0$ considere la EDO

$$y'' + y = e^{-ax} 1_{x \geq b}$$

Encuentre la solución en su forma integral sin transformadas.

Aplicaremos Transformada

$$(i\nu)^2 \mathcal{F}(y)(\nu) + \mathcal{F}(y)(\nu) = \mathcal{F}(e^{-ax} \mathbb{1}_{x\geq b})$$

$$(1-\nu^2) \mathcal{F}(y)(\nu) = \mathcal{F}(e^{-ax} \mathbb{1}_{x\geq b})(\nu)$$

$$\mathcal{F}(e^{-ax} \mathbb{1}_{x\geq 0})(\nu) = \frac{1}{a+i\nu}$$

$$\widehat{f(x - x_0)}(s) = e^{-isx_0} \hat{f}(s)$$

$x \geq b$
$x - b \geq 0$

$$\mathcal{F}(e^{-a(x-b)} \mathbb{1}_{x-b\geq 0})(\nu) = \frac{e^{-i\nu b}}{a+i\nu}$$

$$e^{ab} \mathcal{F}(e^{-ax} \mathbb{1}_{x\geq b})(\nu) = \frac{e^{-i\nu b}}{a+i\nu}$$

$$\mathcal{F}\left(e^{-ax}\mathbb{1}_{x\geq b}\right)(\nu) = \frac{e^{-b(i\nu+a)}}{i\nu+a}$$

$$\Rightarrow \mathcal{F}(y)(\nu) = \left(\frac{1}{1-\nu^2}\right)\left(\frac{e^{-b(i\nu+a)}}{i\nu+a}\right)$$

$$y(x) = \int_{-\infty}^{\infty}\left(\frac{1}{1-\nu^2}\right)\left(\frac{e^{-b(i\nu+a)}}{i\nu+a}\right)e^{i\nu x}\,d\nu$$