**MA2002-3 Cálculo Avanzado y Aplicaciones**
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# Auxiliar 2: Campos Conservativos e Integrales de Trabajo

31 de agosto de 2021

**P1. [Campos Conservativos]**

*a*) Verifique si los siguientes campos son conservativos.

1) $\overrightarrow{F}(x,y,z) = (y^2\cos(x) + z^3)\hat{i} + (2y\sin(x) - 4)\hat{j} + (3xz^2 + 2z)\hat{k}$

2) $\overrightarrow{G}(\rho,\theta,z) = \dfrac{\rho z}{(\rho^2+z^2)^{3/2}}\hat{\rho} - \dfrac{\rho^2}{(\rho^2+z^2)^{3/2}}\hat{k}$

**P2.** Considere que esta en el campo $F(x,y) = (\sin(x)y, sin(y) - cos(x))$ ¿Cuanto es el trabajo realizado por desplazarse por $\Gamma_1 \cup \Gamma_2$ con $\Gamma_1 = (\pi(1-cos(t)), \pi(sin(t)))$, para $t \in [0,\pi)$ y $\Gamma_2 = (\pi(2-\sin(t)), \pi(cos(t)-1)))$ para $t \in [\pi, 2\pi]$

**P3.** *a*) Calcule el gradiente en coordenadas cartesianas de

$$f(x,y,z) = \frac{z}{x^2+y^2+z^2}$$

¿Calcúlelo en coordenadas esféricas? ¿Como se relacionan estos cálculos?

*b*) Se definen las coordenadas parabólicas $(\epsilon, \eta, \phi)$, tal que $x = \epsilon\eta\cos(\phi)$, $y = \epsilon\eta\sin(\phi)$ y $z = \dfrac{1}{2}(\eta^2 - \epsilon^2)$. Calcule el gradiente y el laplaciano en estas coordenadas. ($\eta, \epsilon > 0$ y $\phi \in [0, 2\pi]$)

**P4.** Sea $\psi$ un campo escalar y $F$, $G$ campos vectoriales suficientemente diferenciables. Demuestre las siguientes identidades:

*a*) $div(\psi\nabla\psi) = \|\nabla\psi\|^2 + \psi\Delta\psi$

*b*) $\Delta G = \nabla(div(G)) - rot(rot(G))$

# Propuestos

**P1.** Coordenadas bipolares $(\sigma, \tau)$, con $\sigma \in (0, 2\pi)$ y $\tau \in (-\infty, \infty)$.La $a$ es una constante, estas coordenadas cumplen que:

$$x = a\frac{\sinh(\tau)}{\cosh(\tau) - \cos(\sigma)} \text{ e } y = a\frac{\sin(\sigma)}{\cosh(\tau) - \cos(\sigma)}$$

*a*) Pruebe que el gradiente en estas coordenadas es: $\nabla f = \dfrac{(\cosh(\tau) - \cos(\sigma))}{a}\left(\frac{\partial f}{\partial \sigma}\hat{\sigma} + \frac{\partial f}{\partial \tau}\hat{\tau}\right)$

*b*) Pruebe que la formula para el laplaciano es: $\Delta f = \dfrac{(\cosh(\tau) - \cos(\sigma))^2}{a^2}\left(\frac{\partial^2 f}{\partial \sigma^2} + \frac{\partial^2 f}{\partial \tau^2}\right)$

**Indicación:** Pruebe que $h_\sigma = h_\tau = \dfrac{a}{\cosh(\tau) - \cos(\sigma)}$

**P2.** Pruebe las siguientes identidades:

*a*) $\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \Delta \nabla g$

*b*) $rot(F) = Fdiv(G) - Gdiv(F) + (G \cdot \nabla)F - (F \cdot \nabla)G$

## Resumen

**Integral de trabajo de un campo $\overrightarrow{F}$ sobre una curva**:
Sea $\Gamma$ una curva simple y regular en $\mathbb{R}^3$, y sea $\overrightarrow{F} : \mathbb{R}^3 \to \mathbb{R}^3$ una función continua. Se define la integral de $F$ sobre la curva $\Gamma$, donde $\overrightarrow{r} : [a, b] \to \mathbb{R}^3$ es una parametrización regular de $\Gamma$. mediante:

$$W = \int_\Gamma \overrightarrow{F} \cdot d\overrightarrow{r} := \int_a^b \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \frac{d\overrightarrow{r}(t)}{dt} dt$$

**Coordenadas Cilíndricas**: $T(\rho, \theta, k) = (\rho\cos(\theta), \rho\sin(\theta), k)$ y los factores $h_\rho = 1$, $h_\theta = \rho$ y $h_k = 1$

**Coordenadas Esféricas**: $T(r, \phi, \theta) = (r\cos(\theta)\sin(\phi), r\sin(\theta)\sin(\phi), r\cos(\phi))$ y los factores $h_r = 1$, $h_\theta = r\sin(\phi)$ y $h_\phi = r$.

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{v} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{w}$$

$$\Delta\phi = \frac{1}{h_1h_2h_3}\sum_{j=1}^{3}\frac{\partial}{\partial w_j}\left[\frac{h_1h_2h_3}{h_j^2}\left(\frac{\partial\phi}{\partial w_j}\right)\right]$$

**P1. [Campos Conservativos]**

*a*) Verifique si los siguientes campos son conservativos.

1) $\overrightarrow{F}(x,y,z) = (y^2\cos(x) + z^3)\hat{\imath} + (2y\sin(x) - 4)\hat{\jmath} + (3xz^2 + 2z)\hat{k}$

2) $\overrightarrow{G}(\rho,\theta,z) = \dfrac{\rho z}{(\rho^2+z^2)^{3/2}}\hat{\rho} - \dfrac{\rho^2}{(\rho^2+z^2)^{3/2}}\hat{k}$

$$\boxed{\exists g \text{ Tal que } \nabla g = F}$$

$$\frac{\partial g}{\partial x} = y^2\cos(x) + z^3 \qquad (1)$$

$$\frac{\partial g}{\partial y} = 2y\sin(x) - 4 \qquad (2)$$

$$\frac{\partial g}{\partial z} = 3xz^2 + 2z \qquad (3)$$

$$(1) \Rightarrow \tilde{g}(x,y,z) = \int y^2\cos(x) + z^3\,dx + C(y,z) \quad \text{(No olvidar)}$$

$$= y^2\sin(x) + z^3x + C(y,z)$$

$$\frac{\partial \tilde{g}}{\partial y} = 2y\sin(x) + \frac{\partial C(y,z)}{\partial y}$$

imponer (2) $\Rightarrow 2y\sin(x) - 4 = \frac{\partial \tilde{g}}{\partial y}$

$$\Rightarrow \frac{\partial C(y,z)}{\partial y} = -4 \quad / \int dy$$

$$C(y,z) = -4y + C(z)$$

Reemplazamos

$$\tilde{g}(x,y,z) = y^2\sin(x) + z^3x - 4y + C(z)$$

$$\frac{\partial \tilde{g}}{\partial z} = 3z^2x + \frac{\partial C(z)}{\partial z} \overset{(3)}{=} 3xz^2 + 2z$$

$$\Rightarrow \frac{\partial C(z)}{\partial z} = 2z \Rightarrow C(z) = z^2 + C$$

$$g(x,y,z) = y^2\sin(x) + z^3x - 4y + z^2 + C$$

2) $\overrightarrow{G}(\rho,\theta,z) = \frac{\rho z}{(\rho^2+z^2)^{3/2}}\hat{\rho} - \frac{\rho^2}{(\rho^2+z^2)^{3/2}}\hat{k}$ $+ \frac{\rho}{5}\theta\,\hat{\theta}$

Obs : $\nabla f$ en cilindricas $= \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{k}$

$$\frac{1}{\cancel{\rho}}\frac{\partial f}{\partial \theta} = \frac{\rho^{\cancel{2}}\theta}{5}$$

2) $\overrightarrow{G}(\rho,\theta,z) = \frac{\rho z}{(\rho^2+z^2)^{3/2}}\hat{\rho} - \frac{\rho^2}{(\rho^2+z^2)^{3/2}}\hat{k}$

$$\frac{\partial f}{\partial \rho} = \frac{\rho z}{(\rho^2+z^2)^{\frac{3}{2}}} \quad (1)$$

$$\frac{1}{\rho}\frac{\partial f}{\partial \theta} = 0 \Rightarrow \frac{\partial f}{\partial \theta} = 0 \quad (2)$$

$$\frac{\partial f}{\partial z} = -\frac{\rho^2}{(\rho^2+z^2)^{3/2}} \quad (3)$$

$$(3) \Rightarrow \cancel{\int \frac{\rho^2}{(\rho^2+z^2)^{\frac{3}{2}}}\, dz}$$

$$(1) \Rightarrow \underbrace{\int \frac{z}{(\rho^2+z^2)^{\frac{3}{2}}}\, \rho\, d\rho} + C(0,z) = \tilde{g}$$

$$C.V = u = \rho^2 + z^2$$

$$\frac{du}{2} = \cancel{2}\rho\, d\rho$$

$$\frac{z}{2}\int u^{-\frac{3}{2}}\, du + C(0,z) = \tilde{g}$$

$$\frac{z}{2}(-2)u^{-\frac{1}{2}} + C(0,z) = \tilde{g}$$

Reemplazando

$$\tilde{g} = \frac{-z}{\sqrt{\rho^2+z^2}} + C(\theta, z)$$

(2) $$\frac{\partial \tilde{g}}{\partial \theta} = 0 = \frac{\partial C(\theta, z)}{\partial \theta}$$

$$C(\theta, z) = C(z) \Rightarrow \tilde{g} = \frac{-z}{\sqrt{\rho^2+z^2}} + C(z)$$

Usar (3)

$$\frac{\partial g}{\partial z} = \frac{-\rho^2}{(\rho^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial \tilde{g}}{\partial z} = -\left( \frac{1(\sqrt{\rho^2+z^2} - \frac{z^2}{\sqrt{\rho^2+z^2}}}{\rho^2+z^2} \right) + \frac{\partial C(z)}{\partial z}$$

$$\frac{\partial \tilde{g}}{\partial z} = -\left(\frac{\rho^2+z^2-z^2}{(\rho^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial C(z)}{\partial z}$$

$$= -\frac{\rho^2}{(\rho^2+z^2)^{\frac{3}{2}}} + \frac{\partial C(z)}{\partial z}$$

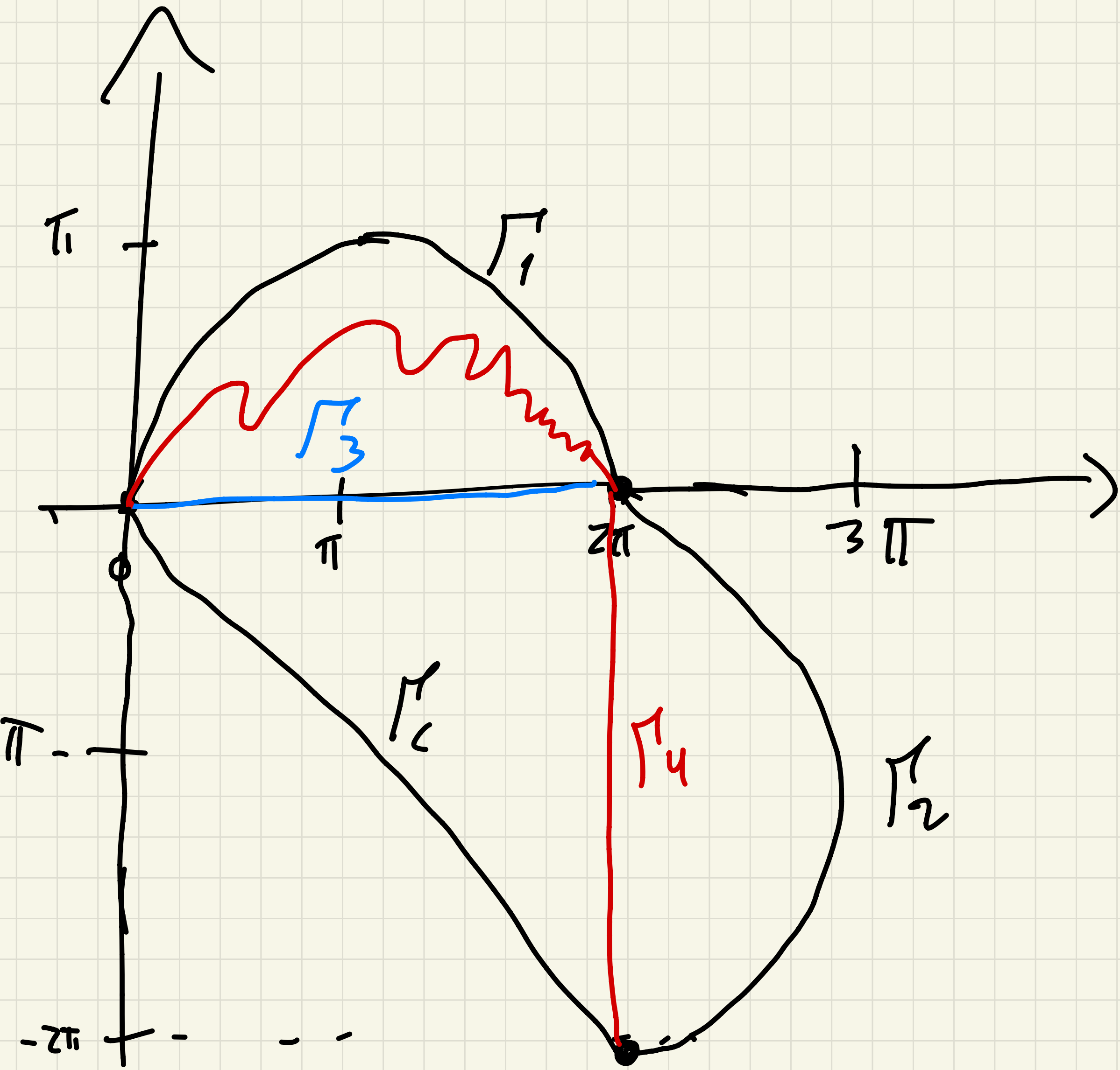
Como quiero que se cumpla (2)

$$\Rightarrow \frac{\partial C(z)}{\partial z} = 0 \Rightarrow C(z) = C$$

$$\boxed{g = -\frac{z}{\sqrt{\rho^2+z^2}}}$$ Cumple Todo

**P2.** Considere que esta en el campo $F(x,y) = (\sin(x)y, sin(y) - cos(x))$ ¿Cuanto es el trabajo realizado por desplazarse por $\Gamma_1 \cup \Gamma_2$ con $\Gamma_1 = (\pi(1-cos(t)), \pi(sin(t)))$, para $t \in [0,\pi)$ y $\Gamma_2 = (\pi(2-\sin(t)), \pi(cos(t)-1)))$ para $t \in [\pi, 2\pi]$

$(-(\cos(t)-1))$




$$W = \int_{\Gamma_1} F(\Gamma(t)) \cdot \frac{d\Gamma}{dt} + \int_{\Gamma_2} F(\Gamma(t)) \frac{d\Gamma}{dt} =$$

$$= \int_0^{\pi} \left( \sin\left(\pi(1-\cos(t))\right)\left(\pi \sin(t)\right), \sin\left(\pi \sin(t)\right) \cdot \cos\left(\pi(1-\cos(t))\right) \right) \cdot \frac{d\Gamma}{dt} \, dt$$

Si: $F$ es conservativo

Puedo decir que $W = \int_{\Gamma_1} F = \int_{\Gamma_3} F$

$$\tilde{\Gamma}(t) = (2t, 0) \quad t \in [0, \pi]$$

$$\Gamma_3 \Rightarrow \frac{d\Gamma}{dt} = (2, 0)$$

$$W_1 = \int_{\Gamma_1} = \int_{\Gamma_3} = \int_0^{\pi} (\mathrm{Sin}(2t)\cdot 0,\ \mathrm{Sin}(0) - \mathrm{Cos}(2t)) \cdot (2,0)$$

$$= \int_0^{\pi} 0 \ dt = 0$$

$$F(x,y) = (\sin(x)y, sin(y) - cos(x))$$

$$\boxed{g(x,y) = -\mathrm{Cos}(x)y - \mathrm{Cos}(y)}$$ (Verificado)

$\Rightarrow F$ conservativa

$$\vec{\Gamma_c} = (t, -t) \qquad t \in [0, 2\pi]$$

$$\frac{d\vec{\Gamma_c}}{dt} = (1, -1)$$

$$W = \int_0^{2\pi} \left(\sin(t)\,t,\ -\sin(t) - \cos(t)\right) \cdot (1,-1)$$

$$= \int_0^{2\pi} \underbrace{-\sin(t)\cdot t}\ + \cancel{\sin(t)} + \cancel{\cos(t)}\ dt$$

$$= \boxed{-2\pi}$$

(Por partes usar
$dv = -\sin(t)\,dt$
$u = t$)

$$\boxed{W = -2\pi}$$

**P3.** *a*) Calcule el gradiente en coordenadas cartesianas de

$$f(x,y,z) = \frac{z}{x^2+y^2+z^2}$$

¿Calcúlelo en coordenadas esféricas? ¿Como se relacionan estos cálculos?

*b*) Se definen las coordenadas parabólicas $(\epsilon, \eta, \phi)$, tal que $x = \epsilon\eta\cos(\phi)$, $y = \epsilon\eta\sin(\phi)$ y $z = \frac{1}{2}(\eta^2 - \epsilon^2)$. Calcule el gradiente y el laplaciano en estas coordenadas. ($\eta, \epsilon > 0$ y $\phi \in [0, 2\pi]$)

a) $$\nabla f = \left( \frac{-2xz}{(x^2+y^2+z^2)^2}, \frac{-2yz}{(x^2+y^2+z^2)^2}, \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2} \right)$$

$$x = r\cos\theta \sin\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos(\phi)$$

$$f(r,\theta,\phi) = \frac{r\cos\phi}{r^2} = \frac{\cos\phi}{r}$$

$$\nabla g(r,\theta,\phi) = \frac{\partial g}{\partial r}\hat{r} + \frac{1}{r\sin\phi}\frac{\partial g}{\partial \theta}\hat{\theta} + \frac{1}{r}\frac{\partial g}{\partial \phi}\hat{\phi}$$

$$\nabla f(r,\theta,\phi) = \frac{-r^2\cos\phi}{r^4}\hat{r} - \frac{r^2\sin\phi}{r^4}\hat{\sigma}$$

$$\hat{r} = (\cos\theta\,\hat{\theta} + \sin\theta\,\hat{\phi})$$

(Mal idea No más)

b) $r(\xi, \eta, \phi) \to (x, y, z)$

$$J_r = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \phi} \\[2ex] \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial y}{\partial \phi} \\[2ex] \dfrac{\partial z}{\partial \xi} & \dfrac{\partial z}{\partial \eta} & \dfrac{\partial z}{\partial \phi} \end{bmatrix}$$

e $x = \epsilon\eta\cos(\phi)$, $y = \epsilon\eta\sin(\phi)$ y $z = \frac{1}{2}(\eta^2 - \epsilon^2)$.

das $(\eta, \epsilon > 0$ y $\phi \in [0, 2\pi])$

$$= \begin{bmatrix} \eta\cos(\phi) & \xi\cos(\phi) & -\xi\eta\sin\phi \\ \eta\sin(\phi) & \xi\sin(\phi) & \xi\eta\cos\phi \\ -\xi & \eta & 0 \end{bmatrix}$$

$$\left[ \vec{\xi} \;\vdots\; \vec{\eta} \;\vdots\; \vec{\phi} \right]$$

$$h_\xi = \|\vec{\xi}\| = \sqrt{\eta^2 + \xi^2}$$

$$h_\eta = \|\vec{\eta}\| = \sqrt{\eta^2 + \xi^2}$$

$$h_\phi = \|\vec{\phi}\| = \xi\eta$$

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{v} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{w}$$

$$\Delta\phi = \frac{1}{h_1 h_2 h_3}\sum_{j=1}^{3}\frac{\partial}{\partial w_j}\left[\frac{h_1 h_2 h_3}{h_j^2}\left(\frac{\partial \phi}{\partial w_j}\right)\right]$$

$$\nabla f = \frac{1}{\sqrt{\eta^2+\xi^2}}\left(\frac{\partial f}{\partial \xi}\hat{\xi} + \frac{\partial f}{\partial \eta}\hat{\eta}\right) + \frac{1}{\xi\eta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

$$\Delta f = \frac{1}{(\eta^2+\xi^2)\,\xi\eta}\left[\frac{\partial}{\partial \xi}\left((\xi\eta)\frac{\partial f}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left((\xi\eta)\frac{\partial f}{\partial \eta}\right) + \frac{\partial}{\partial \phi}\left(\frac{(\eta^2+\xi^2)}{\xi\eta}\frac{\partial f}{\partial \phi}\right)\right]$$

**P4.** Sea $\psi$ un campo escalar y $F$, $G$ campos vectoriales suficientemente diferenciables. Demuestre las siguientes identidades:

a) $div(\psi\nabla\psi) = \|\nabla\psi\|^2 + \psi\Delta\psi$

b) $\Delta G = \nabla(div(G)) - rot(rot(G))$

$$\Delta G = \left(\frac{\partial^2 G_1}{\partial x^2} + \frac{\partial^2 G_1}{\partial y^2} + \frac{\partial^2 G_1}{\partial z^2}\right)\hat{\imath}$$

$$+ \left(\frac{\partial^2 G_2}{\partial x^2} + \frac{\partial^2 G_2}{\partial y^2} + \frac{\partial^2 G_2}{\partial z^2}\right)\hat{\jmath}$$

$$+ \left(\frac{\partial^2 G_3}{\partial x^2} + \frac{\partial^2 G_3}{\partial y^2} + \frac{\partial^2 G_3}{\partial z^2}\right)\hat{k}$$

a) $\psi\nabla\psi = \psi\frac{\partial\psi}{\partial x}\hat{\imath} + \psi\frac{\partial\psi}{\partial y}\hat{\jmath} + \psi\frac{\partial\psi}{\partial z}\hat{k}$

$$div(\psi\nabla\psi) = \frac{\partial}{\partial x}\left(\psi\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\psi\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\psi\frac{\partial\psi}{\partial z}\right)$$

$$= \left(\frac{\partial \varphi}{\partial x}\right)^2 + \varphi \frac{\partial^2 \varphi}{\partial x^2} + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \varphi \frac{\partial^2 \varphi}{\partial y^2}$$

$$+ \left(\frac{\partial \varphi}{\partial z}\right)^2 + \varphi \left(\frac{\partial^2 \varphi}{\partial z^2}\right)$$

$$= \|\nabla \varphi\|^2 + \varphi \, \Delta \varphi$$

b)

$$\Delta G = \nabla(div(G)) - rot(rot(G))$$

$$\operatorname{rot}(G) = \left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right)\hat{i} + \left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right)\hat{j} + \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\hat{k}$$

$$\operatorname{rot}(\operatorname{rot}(G)) = \left(\frac{\partial}{\partial y}\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right)\right)\hat{i}$$

$$+ \left(\frac{\partial}{\partial z}\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right)\right)\hat{k}$$

$$= \left(-\frac{\partial^2 G_1}{\partial y^2} - \frac{\partial^2 G_1}{\partial z^2} + \frac{\partial^2 G_2}{\partial x \partial y} + \frac{\partial^2 G_3}{\partial x \partial z}\right)\hat{i}$$

$$+ \left(-\frac{\partial^2 G_2}{\partial x^2} - \frac{\partial^2 G_2}{\partial z^2} + \frac{\partial^2 G_1}{\partial x \partial y} + \frac{\partial^2 G_3}{\partial y \partial z}\right)\hat{j}$$

$$+\left(-\frac{\partial^2 G_3}{\partial x^2}-\frac{\partial^2 G_3}{\partial y^2}+\frac{\partial^2 G_1}{\partial x \partial z}+\frac{\partial^2 G_2}{\partial y \partial z}\right)\hat{k}$$

$$\nabla(\mathrm{Div}(G))=\nabla\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}+\frac{\partial G_3}{\partial z}\right)$$

$$=\left(\frac{\partial^2 G_1}{\partial x^2}+\frac{\partial G_2}{\partial x \partial y}+\frac{\partial G_3}{\partial x \partial z}\right)\hat{i}$$

$$+\left(\frac{\partial^2 G_2}{\partial y^2}+\frac{\partial G_1}{\partial x \partial y}+\frac{\partial G_3}{\partial y \partial z}\right)\hat{j}$$

$$+\left(\frac{\partial^2 G_3}{\partial z^2}+\frac{\partial G_1}{\partial x \partial z}+\frac{\partial G_2}{\partial y \partial z}\right)\hat{k}$$

$$\nabla(\mathrm{Div}(G))-\mathrm{rot}(\mathrm{rot}(G))=\left(\frac{\partial^2 G_1}{\partial x^2}+\frac{\partial^2 G_1}{\partial y^2}+\frac{\partial^2 G_1}{\partial z^2}\right)\hat{i}$$

$$+\left(\frac{\partial^2 G_2}{\partial x^2}+\frac{\partial^2 G_2}{\partial y^2}+\frac{\partial^2 G_2}{\partial z^2}\right)\hat{j}$$

$$+\left(\frac{\partial^2 G_3}{\partial x^2}+\frac{\partial^2 G_3}{\partial y^2}+\frac{\partial^2 G_3}{\partial z^2}\right)\hat{k}$$

**P1.** Coordenadas bipolares $(\sigma, \tau)$, con $\sigma \in (0, 2\pi)$ y $\tau \in (-\infty, \infty)$.La $a$ es una constante, estas coordenadas cumplen que:

$$x = a\frac{\sinh(\tau)}{\cosh(\tau) - \cos(\sigma)} \text{ e } y = a\frac{\sin(\sigma)}{\cosh(\tau) - \cos(\sigma)}$$

*a*) Pruebe que el gradiente en estas coordenadas es: $\nabla f = \dfrac{(\cosh(\tau) - \cos(\sigma))}{a} \left(\frac{\partial f}{\partial \sigma}\hat{\sigma} + \frac{\partial f}{\partial \tau}\hat{\tau}\right)$

*b*) Pruebe que la formula para el laplaciano es: $\Delta f = \dfrac{(\cosh(\tau) - \cos(\sigma))^2}{a^2} \left(\frac{\partial^2 f}{\partial \sigma^2} + \frac{\partial^2 f}{\partial \tau^2}\right)$

**Indicación:** Pruebe que $h_\sigma = h_\tau = \dfrac{a}{\cosh(\tau) - \cos(\sigma)}$

**P2.** Pruebe las siguientes identidades:

$$\frac{\partial x}{\partial \sigma} = \frac{a \sin(\tau) \sin\sigma}{(\cosh(\tau) - \cos(\sigma))^2}$$

$$\frac{\partial y}{\partial \sigma} = a\left(\frac{\cos\sigma(\cosh(\tau) - \cos\sigma) - \sin\sigma^2}{(\cosh(\tau) - \cos(\sigma))^2}\right).$$

$$= \frac{a(\cosh(\tau)\cos\sigma - 1)}{(\cosh\tau - \cos(\sigma))^2}$$

$$h_\sigma^2 = \frac{a^2}{(\cosh(\tau) - \cos\sigma)^4}\left(\sin(\tau)^2 \sin\sigma^2 + \cosh(\tau)^2 \cos\sigma^2 - 2\cosh\tau\cos\sigma + 1\right)$$

$$\cosh(\tau)^2 - 2\cosh\tau\cos\sigma + \cos\sigma^2$$ [illegible]

$$= \frac{a^2}{(\ \ )^4}\left((\cosh\tau^2-1)(1-\cos\sigma^2) + \cancel{\cosh(\tau)^2\cos(\sigma)^2} - 2\cosh\tau\cos\sigma \;\cancel{+1}\right)$$

$$= \frac{a^2}{(\ \ )^4}\left(\cancel{-1}+\cosh\tau^2+\cos\sigma^2 - \cancel{\cosh\tau^2\cos\sigma^2} + \cosh\tau^2\cos\sigma^2 - 2\cosh\tau\cos\sigma + \cancel{1}\right)$$

$$= \frac{a^2}{(\ \ )^4}(\ \ )^2 = \frac{a^2}{(\ \ )^2}$$

$$h_\sigma = \frac{a}{(\cosh\tau - \cos\sigma)}$$

$$h_\tau = \frac{a}{(\cosh\tau - \cos\sigma)}$$

Caso 2 Variables

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{v} +$$ [redacted]

$$\Delta\phi = \frac{1}{h_1 h_2}\sum_{j=1}^{2}\frac{\partial}{\partial w_j}\left[\frac{h_1 h_2}{h_j^2}\left(\frac{\partial \phi}{\partial w_j}\right)\right]$$

**P2.** Pruebe las siguientes identidades:

a) $\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \cdot \nabla g$

b) $rot(F) = Fdiv(G) - Gdiv(F) + (G\cdot\nabla)F - (F\cdot\nabla)G$

$$\Delta(fg) = \frac{\partial^2}{\partial x^2}(fg) + \frac{\partial^2}{\partial y^2}(fg) + \frac{\partial^2}{\partial z^2}(fg)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\cdot g + \frac{\partial g}{\partial x} f\right) + \ldots$$

$$= \frac{\partial^2 f}{\partial x^2} g + 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial^2 g}{\partial x^2}\cdot f$$

$$+ \frac{\partial^2 f}{\partial y^2} g + 2\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial^2 g}{\partial y^2}\cdot f$$

$$+ \frac{\partial^2 f}{\partial z^2} g + 2\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + \frac{\partial^2 g}{\partial z^2}\cdot f$$

$$\nabla f \cdot \nabla g = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix}$$

b) $(F \cdot \nabla) G = F \cdot (\nabla G_1)\hat{\imath} + F \cdot (\nabla G_2)\hat{\jmath}$

$+ F \cdot (\nabla G_3)\hat{k}$

**P2.** Pruebe las siguientes identidades:

*a)* $\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \Delta \nabla g$

*b)* $rot(F) = F div(G) - G div(F) + (G \cdot \nabla)F - (F \cdot \nabla)G$

$$rot(F) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath}$$

$$+ \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\jmath}$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$(G\cdot\nabla)F = \left(G_1\cdot\frac{\partial F_1}{\partial x} + G_2\cdot\frac{\partial F_1}{\partial y} + G_3\frac{\partial F_1}{\partial z}\right)\hat{\imath}$$

$$+\left(G_1\cdot\frac{\partial F_2}{\partial x} + G_2\cdot\frac{\partial F_2}{\partial y} + G_3\frac{\partial F_2}{\partial z}\right)\hat{\jmath}$$

$$+\left(G_1\cdot\frac{\partial F_3}{\partial x} + G_2\cdot\frac{\partial F_3}{\partial y} + G_3\frac{\partial F_3}{\partial z}\right)\hat{k}$$

(Ignorarla esta nota as an)

$$\operatorname{rot}(F\times G)$$