INVESTMENT SCIENCE

11 MODELS OF ASSET DYNAMICS

True multiperiod investments fluctuate in value, distribute random dividends, exist in an environment of variable interest rates, and are subject to a continuing variety of other uncertainties. This chapter initiates the study of such investments by showing how to model asset price fluctuations conveniently and realistically. This chapter therefore contains no investment principles as such. Rather it introduces the mathematical models that form the foundation for the analyses developed in later chapters.

Two primary model types are used to represent asset dynamics: binomial lattices and Ito processes. Binomial lattices are analytically simpler than Ito processes, and they provide an excellent basis for computational work associated with investment problems. For these reasons it is best to study binomial lattice models first. The important investment concepts can all be expressed in terms of these models, and many real investment problems can be formulated and solved using the binomial lattice framework. Indeed, roughly 80% of the material in later chapters is presented in terms of binomial lattice models

Ito processes are more realistic than binomial lattice models in the sense that they have a continuum of possible stock prices at each period, not just two. Ito process models also allow some problems to be solved analytically, as well as computationally. They also provide the foundation for constructing binomial lattice models in a clear and consistent manner. For these reasons Ito process models are fundamental to dynamic problems. For a complete understanding of investment principles, it is important to understand these models.

The organization of this chapter is based on the preceding viewpoint concerning the roles of different models. The first section presents the binomial lattice model directly. With this background most of the material in later chapters can be studied. Therefore you may wish to read only this first section and then skip to the next chapter.

The remaining sections consider models that have a continuum of price values. These models are developed progressively from discrete-time models to continuous-time models based on Ito processes.

11.1 BINOMIAL LATTICE MODEL

To define a binomial lattice model, a basic period length is established (such as 1 week). According to the model, if the price is known at the beginning of a period, the price at the beginning of the next period is one of only two possible values. Usually these two possibilities are defined to be multiples of the price at the previous period—a multiple u (for up) and a multiple d (for down). Both u and d are positive, with $u > 1$ and (usually) $d < 1$. Hence if the price at the beginning of a period is S, it will be either uS or dS at the next period. The probabilities of these possibilities are p and $1 - p$, respectively, for some given probability p, $0 < p < 1$. That is, if the current price is S, there is a probability p that the new price will be uS and a probability $1 - p$ that it will be dS. This model continues on for several periods.

The general form of such a lattice is shown in Figure 11.1. The stock price can be visualized as moving from node to node in a rightward direction. The probability of an upward movement from any node is p and the probability of a downward movement is $1 - p$. A lattice is the appropriate structure in this case, rather than a tree, because an up movement followed by a down is identical to a down followed by an up. Both produce ud times the price.

The model may at first seem too simple because it permits only two possible values at the next period. But if the period length is small, many values are possible after several short steps.

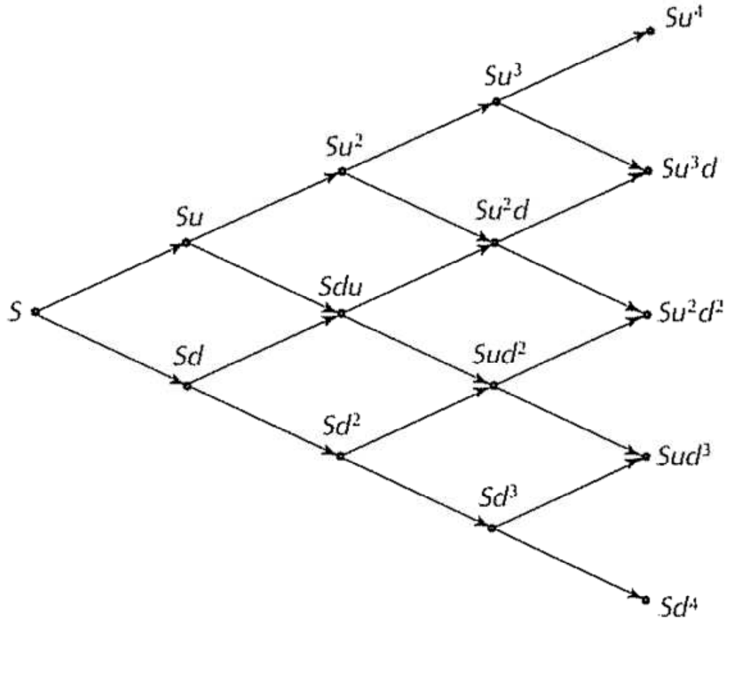

FIGURE 11.1 Binomial lattice stock model. At each step the stock price S either increases to uS or decreases to dS.

To specify the model completely, we must select values for u and d and the probability p. These should be chosen in such a way that the true stochastic nature of the stock is captured as faithfully as possible, as will be discussed.

Because the model is multiplicative in nature (the new value being uS or dS, with $u > 0$, $d > 0$), the price will never become negative. It is therefore possible to consider the logarithm of price as a fundamental variable. For reasons discussed in later sections, use of the logarithm is in fact very helpful and leads to simple formulas for selecting the parameters.

Accordingly, we define ν as the expected yearly growth rate.[1] Specifically,

$$\nu = \mathrm{E}\left[\ln(S_T/S_0)\right]$$

where S_0 is the initial stock price and S_T is the price at the end of 1 year.

Likewise, we define σ as the yearly standard deviation. Specifically,

$$\sigma^2 = \mathrm{var}\left[\ln(S_T/S_0)\right].$$

If a period length of Δt is chosen, which is small compared to 1, the parameters of the binomial lattice can be selected as

$$\begin{aligned} p &= \frac{1}{2} + \frac{1}{2}\left(\frac{\nu}{\sigma}\right)\sqrt{\Delta t} \\ u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}}. \end{aligned} \tag{11.1}$$

With this choice, the binomial model will closely match the values of ν and σ (as shown later); that is, the expected growth rate of $\ln S$ in the binomial model will be nearly ν, and the variance of that rate will be nearly σ^2. The closeness of the match improves if Δt is made smaller, becoming exact as Δt goes to zero.

Example 11.1 (A volatile stock) Consider a stock with the parameters $\nu = 15\%$ and $\sigma = 30\%$. We wish to make a binomial model based on weekly periods. According to (11.1), we set

$$u = e^{.30/\sqrt{52}} = 1.04248, \qquad d = 1/u = .95925$$

and

$$p = \frac{1}{2}\left(1 + \frac{.15}{.30}\sqrt{\frac{1}{52}}\right) = .534669.$$

The lattice for this example is shown in Figure 11.2, assuming $S(0) = 100$.

We shall return to the binomial lattice later in this chapter after studying models that allow a continuum of prices. The binomial model will be found to be a natural approximation to these models.

[1] If the process were deterministic, then $\nu = \ln(S_T/S_0)$ implies $S_T = S_0e^{\nu T}$, which shows that ν is the exponential growth rate.

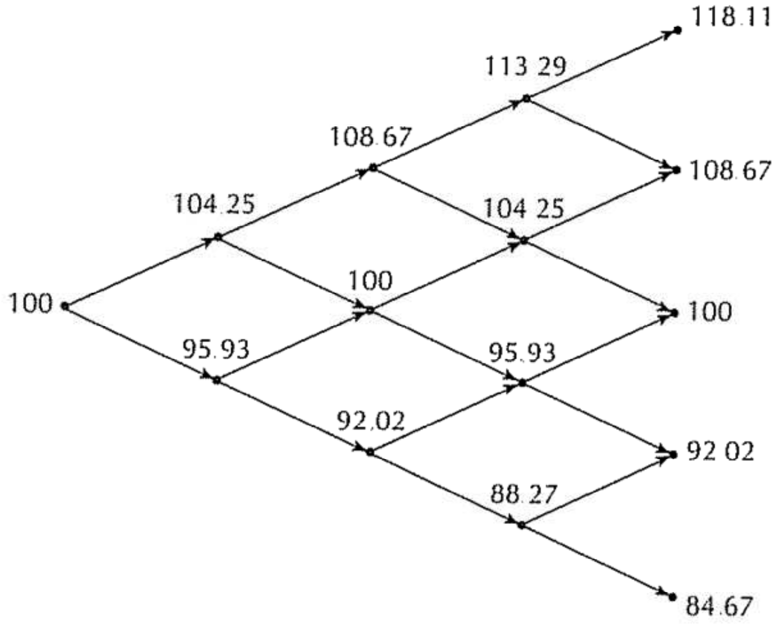

FIGURE 11.2 Lattice for Example 11.1. The parameters are chosen so that the expected growth rate of the logarithm of price and the variance of that growth rate match the known corresponding values for the asset.

11.2 THE ADDITIVE MODEL

We now study models with the property that price can range over a continuum. First we shall consider discrete-time models, beginning with the additive model of this section, and then later we shall consider continuous-time models defined by Ito processes.

Let us focus on $N+1$ time points, indexed by k, $k = 0, 1, 2, \ldots, N$. We also focus on a particular asset that is characterized by a price at each time. The price at time k is denoted by $S(k)$. Our model will recognize that the price in any one time is dependent to some extent on previous prices.

The simplest model is the **additive model,**

$$S(k+1) = aS(k) + u(k) \tag{11.2}$$

for $k = 0, 1, 2, \ldots, N$. In this equation a is a constant (usually $a > 1$) and the quantities $u(k)$, $k = 0, 1, \ldots, N-1$, are random variables. The $u(k)$'s can be thought of as "shocks" or "disturbances" that cause the price to fluctuate. To operate or run this model, an initial price $S(0)$ is specified; then once the random variable $u(0)$ is given, $S(1)$ can be determined. The process then repeats progressively in a stepwise fashion, determining $S(2), S(3), \ldots, S(N)$.

The key ingredient of this model is the sequence of random variables $u(k)$, $k = 1, 2, \ldots, N$. We assume that these are mutually statistically independent.

Note that the price at any time depends only on the price at the most recent previous time and the random disturbance. It does not explicitly depend on other previous prices.

Normal Price Distribution

It is instructive to solve explicitly for a few of the prices from (11.2). By direct substitution we have

$$\begin{aligned} S(1) &= aS(0) + u(0) \\ S(2) &= aS(1) + u(1) \\ &= a^2 S(0) + au(0) + u(1). \end{aligned}$$

By simple induction it can be seen that for general k,

$$S(k) = a^k S(0) + a^{k-1}u(0) + a^{k-2}u(1) + \cdots + u(k-1). \quad (11.3)$$

Hence $S(k)$ is $a^k S(0)$ plus the sum of k random variables.

Frequently we assume that the random variables $u(k)$, $k = 0, 1, 2, \ldots, N-1$, are independent normal random variables with a common variance σ^2. Then, since a linear combination of normal random variables is also normal (see Appendix A), it follows from (11.3) that $S(k)$ is itself a normal random variable.

If the expected values of all the $u(k)$'s are zero, then the expected value of $S(k)$ is

$$\mathrm{E}[S(k)] = a^k S(0).$$

When $a > 1$, this model has the property that the expected value of the price increases geometrically (that is, according to a^k). Indeed, the constant a is the growth rate factor of the model.

The additive model is structurally simple and easy to work with. The expected value of price grows geometrically, and all prices are normal random variables. However, the model is seriously flawed because it lacks realism. Normal random variables can take on negative values, which means that the prices in this model might be negative as well; but real stock prices are never negative. Furthermore, if a stock were to begin at a price of, say, \$1 with a σ of, say, \$.50 and then drift upward to a price of \$100, it seems very unlikely that the σ would remain at \$.50. It is more likely that the standard deviation would be proportional to the price. For these reasons the additive model is not a good general model of asset dynamics. The model is useful for localized analyses, over short periods of time (perhaps up to a few months for common stocks), and it is a useful building block for other models, but it cannot be used alone as an ongoing model representing long- or intermediate-term fluctuations. For this reason we must consider a better alternative, which is the multiplicative model. (However, our understanding of the additive model will be important for that more advanced model.)

11.3 THE MULTIPLICATIVE MODEL

The **multiplicative model** has the form

$$S(k+1) = u(k)S(k) \quad (11.4)$$

for $k = 0, 1, \ldots, N-1$. Here again the quantities $u(k)$, $k = 0, 1, 2, \ldots, N-1$, are mutually independent random variables. The variable $u(k)$ defines the *relative* change in price between times k and $k+1$. This relative change is $S(k+1)/S(k)$, which is independent of the overall magnitude of $S(k)$. It is also independent of the units of price. For example, if we change units from U.S. dollars to German marks, the relative price change is still $u(k)$.

The multiplicative model takes a familiar form if we take the natural logarithm of both sides of the equation. This yields

$$\ln S(k+1) = \ln S(k) + \ln u(k) \tag{11.5}$$

for $k = 0, 1, 2, \ldots, N-1$. Hence in this form the model is of the additive type with respect to the logarithm of the price, rather than the price itself. Therefore we can use our knowledge of the additive model to analyze the multiplicative model.

It is now natural to specify the random disturbances directly in terms of the $\ln u(k)$'s. In particular we let

$$w(k) = \ln u(k)$$

for $k = 0, 1, 2, \ldots, N-1$, and we specify that these $w(k)$'s be normal random variables. We assume that they are mutually independent and that each has expected value $\overline{w}(k) = \nu$ and variance σ^2.

We can express the original multiplicative disturbances as

$$u(k) = e^{w(k)} \tag{11.6}$$

for $k = 0, 1, 2, \ldots, N-1$. Each of the variables $u(k)$ is said to be a **lognormal** random variable since its logarithm is in fact a normal random variable.

Notice that now there is no problem with negative values. Although the normal variable $w(k)$ may be negative, the corresponding $u(k)$ given by (11.6) is always positive. Since the random factor by which a price is multiplied is $u(k)$, it follows that prices remain positive in this model.

Lognormal Prices

The successive prices of the multiplicative model can be easily found to be

$$S(k) = u(k-1)u(k-2)\cdots u(0)S(0).$$

Taking the natural logarithm of this equation we find

$$\ln S(k) = \ln S(0) + \sum_{i=0}^{k-1} \ln u(i) = \ln S(0) + \sum_{i=0}^{k-1} w(i).$$

The term $\ln S(0)$ is a constant, and the $w(i)$'s are each normal random variables. Since the sum of normal random variables is itself a normal random variable (see Appendix A), it follows that $\ln S(k)$ is normal. In other words, all prices are lognormal under the multiplicative model.

If each $w(i)$ has expected value $\overline{w}(i) = \nu$ and variance σ^2, and all are mutually independent, then we find

$$E[\ln S(k)] = \ln S(0) + \nu k \tag{11.7a}$$

$$\text{var}[\ln S(k)] = k\sigma^2 \tag{11.7b}$$

Hence both the expected value and the variance increase linearly with k.

Real Stock Distributions

At this point it is natural to ask how well this theoretical model fits actual stock price behavior. Are real stock prices lognormal?

The answer is that, based on an analysis of past stock price records, the price distributions of most stocks are actually quite close to lognormal. To verify this, we select a nominal period length of, say, 1 week and record the differences $\ln S(k+1) - \ln S(k)$ for many values of k; that is, we record the weekly changes in the logarithm of the prices for many weeks. We then construct a histogram of these values and compare it with that of a normal distribution of the same variance. Typically, the measured distribution is quite close to being normal, except that the observed distribution often is slightly smaller near the mean and larger at extremely large values (either positive or negative large values). This slight change in shape is picturesquely termed **fat tails.** (See Figure 11.3.[2]) The observed distribution is larger in the tails than a normal

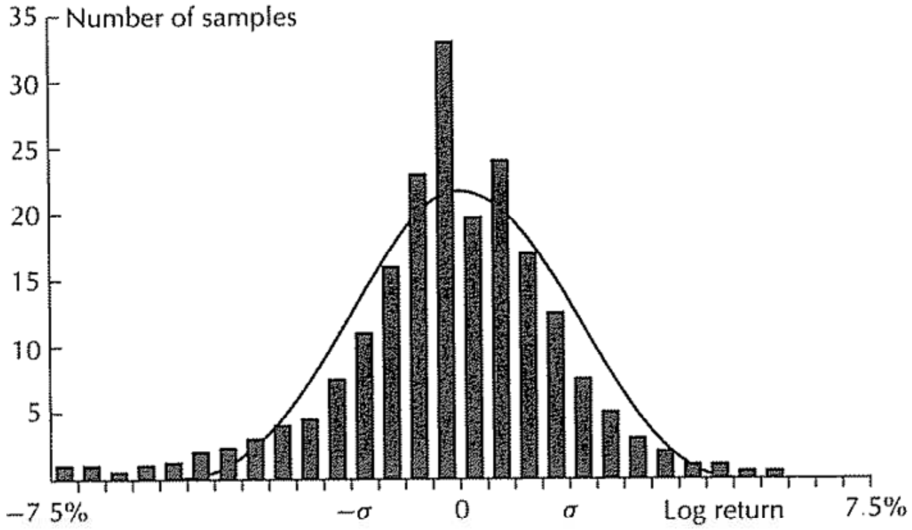

FIGURE 11.3 Observed distribution of the logarithm of return. The distribution has "fatter tails" than a normal distribution of the same variance.

[2]The figure shows a histogram of American Airlines weekly log stock returns for the 10-year period of 1982–1992. Shown superimposed is the normal distribution with the same (sample) mean and standard deviation. Along with fat tails there is invariably a "skinny middle."

distribution. This implies that large price changes tend to occur somewhat more frequently than would be predicted by a normal distribution of the same variance. For most applications (but not all) this slight discrepancy is not important

11.4 TYPICAL PARAMETER VALUES*

The return of a stock over the period between k and $k+1$ is $S(k+1)/S(k)$, which under the multiplicative model is equal to $u(k)$. The value of $w(k) = \ln u(k)$ is therefore the logarithm of the return. The mean value of $w(k)$ is denoted by ν and the variance of $w(k)$ by σ^2. Typical values of these parameters for assets such as common stocks can be inferred from our knowledge of corresponding values for returns. Thus for stocks, typical values of $\nu = \mathrm{E}[w(k)]$ and $\sigma = \mathrm{stdev}\,[w(k)]$ might be

$$\nu = 12\%, \qquad \sigma = 15\%$$

when the length of a period is 1 year. If the period length is less than a year, these values scale downward;[3] that is, if the period length is p part of a year, then

$$\nu_p = p\nu, \qquad \sigma_p = \sqrt{p}\sigma.$$

The values can be estimated from historical records in the standard fashion (but with caution as to the validity of these estimates, as raised in Chapter 8). If we have $N+1$ time points of data, spanning N periods, the estimate of the single-period ν is

$$\begin{aligned}\hat{\nu} &= \frac{1}{N}\sum_{k=0}^{N-1} \ln\left[\frac{S(k+1)}{S(k)}\right] = \frac{1}{N}\sum_{k=0}^{N-1}[\ln S(k+1) - \ln S(k)] \\ &= \frac{1}{N}\ln\left[\frac{S(N)}{S(0)}\right].\end{aligned}$$

Hence all that matters is the ratio of the last to the first price.

The standard estimate of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{N-1}\sum_{k=0}^{N-1}\left\{\ln\left[\frac{S(k+1)}{S(k)}\right] - \hat{\nu}\right\}^2.$$

As with the estimation of return parameters, the error in these estimates can be characterized by their variances. For ν this variance is

$$\mathrm{var}(\hat{\nu}) = \sigma^2/N$$

and for σ^2 it is [assuming $w(k)$ is normal]

$$\mathrm{var}(\hat{\sigma}^2) = 2\sigma^4/(N-1).$$

[3] Using log returns, the scaling is *exactly* proportional. There is no error due to compounding as with returns (without the log). (See Exercise 2.)

Hence for the values assumed earlier, namely, $\nu = .12$ and $\sigma = .15$, we find that 10 years of data is required to reduce the standard deviation of the estimate[4] of ν to .05 (which is still a sizable fraction of the true value). On the other hand, with only 1 year of weekly data we can obtain a fairly good estimate[5] of σ^2.

11.5 LOGNORMAL RANDOM VARIABLES

If u is a lognormal random variable, then the variable $w = \ln u$ is normal. In this case we found that the prices in the multiplicative model are all lognormal random variables. It is therefore useful to study a few important properties of such random variables.

The general shape of the probability distribution of a lognormal random variable is shown in Figure 11.4. Note that the variable is always nonnegative and the distribution is somewhat skewed.

Suppose that w is normal and has expected value $\overline{w}$ and variance σ^2. What is the expected value of $u = e^w$? A quick guess might be $\overline{u} = e^{\overline{w}}$, but this is wrong. Actually $\overline{u}$ is greater than this by the factor $e^{\frac{1}{2}\sigma^2}$; that is,

$$\overline{u} = e^{\overline{w}+\frac{1}{2}\sigma^2}. \tag{11.8}$$

This result can be intuitively understood by noting that as σ is increased, the lognormal distribution will spread out. It cannot spread downward below zero, but it can spread upward unboundedly. Hence the mean value increases as σ increases.

The extra term $\frac{1}{2}\sigma^2$ is actually fairly small for low-volatility stocks. For example, consider a stock with a yearly $\overline{w} = .12$ and a yearly σ of .15. The correction term is

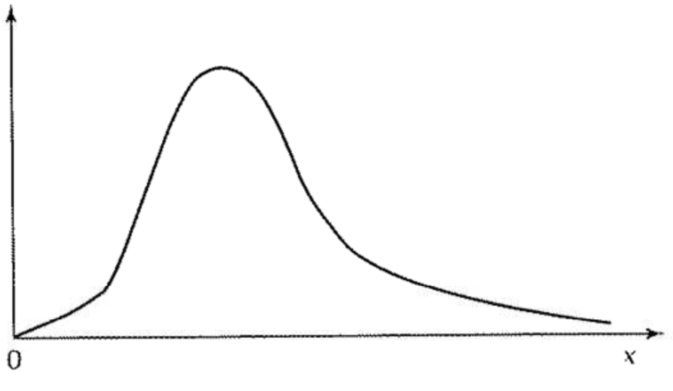

FIGURE 11.4 Lognormal distribution. The lognormal distribution is nonzero only for $x > 0$

[4] $\sigma(\hat{\nu}) = \dfrac{\sigma}{\sqrt{N}} = \dfrac{\sigma}{\sqrt{10}} = \dfrac{.15}{3.16} = .05$

[5] $\mathrm{var}(\hat{\sigma}^2) = (52)^2\,\mathrm{var}(\hat{\sigma}_w^2) = \dfrac{(52)^2 \times 2\sigma_w^4}{N-1} = \dfrac{(52)^2 \times 2\sigma^4}{(52)^2 \times 51} = \dfrac{2\sigma^4}{51}$ Hence $\sigma(\hat{\sigma}^2) = \dfrac{\sqrt{2}\sigma^2}{\sqrt{51}} \approx \dfrac{\sigma^2}{5}$

$\frac{1}{2}\sigma^2 = .0225$, which is small compared to $\overline{w}$. For stocks with high volatility, however, the correction can be significant.

11.6 RANDOM WALKS AND WIENER PROCESSES

In Section 11.7 we will shorten the period length in a multiplicative model and take the limit as this length goes to zero. This will produce a model in continuous time. In preparation for that step, we introduce special random functions of time, called random walks and Wiener processes.

Suppose that we have N periods of length Δt. We define the additive process z by

$$z(t_{k+1}) = z(t_k) + \epsilon(t_k)\sqrt{\Delta t} \tag{11.9}$$

$$t_{k+1} = t_k + \Delta t \tag{11.10}$$

for $k = 0, 1, 2, \ldots, N$. This process is termed a **random walk.** In these equations $\epsilon(t_k)$ is a normal random variable with mean 0 and variance 1—a **standardized normal random variable.** These random variables are mutually uncorrelated; that is, $\mathrm{E}[\epsilon(t_j)\epsilon(t_k)] = 0$ for $j \neq k$. The process is started by setting $z(t_0) = 0$. Thereafter a particular realized path wanders around according to the happenstance of the random variables $\epsilon(t_k)$. [The reason for using $\sqrt{\Delta t}$ in (11.9) will become clear shortly.] A particular path of a random walk is shown in Figure 11.5.

Of special interest are the difference random variables $z(t_k) - z(t_j)$ for $j < k$. We can write such a difference as

$$z(t_k) - z(t_j) = \sum_{i=j}^{k-1} \epsilon(t_i)\sqrt{\Delta t}.$$

This is a normal random variable because it is the sum of normal random variables. We find immediately that

$$\mathrm{E}[z(t_k) - z(t_j)] = 0.$$

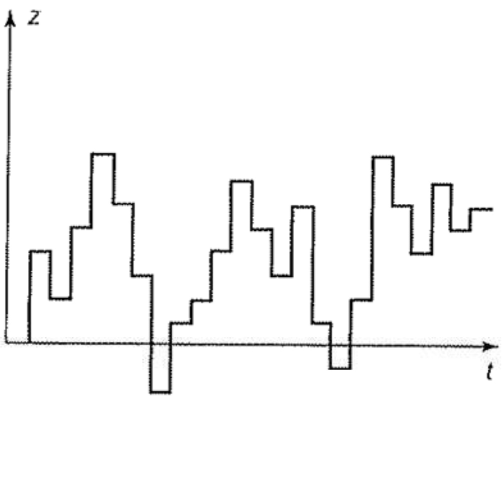

FIGURE 11.5 Possible random walk. The movements are determined by normal random variables.

Also, using the independence of the $\epsilon(t_k)$'s, we find

$$\begin{aligned}\operatorname{var}[z(t_k) - z(t_j)] &= \mathrm{E}\left[\sum_{i=j}^{k-1} \epsilon(t_i)\sqrt{\Delta t}\right]^2 \\ &= \mathrm{E}\left[\sum_{i=j}^{k-1} \epsilon(t_i)^2 \Delta t\right] \\ &= (k-j)\Delta t \;=\; t_k - t_j .\end{aligned}$$

Hence the variance of $z(t_k) - z(t_j)$ is exactly equal to the time difference $t_k - t_j$ between the points. This calculation also shows why $\sqrt{\Delta t}$ was used in the definition of the random walk so that Δt would appear in the variance.

It should be clear that the difference variables associated with two different time intervals are uncorrelated if the two intervals are nonoverlapping. That is, if $t_{k_1} < t_{k_2} \leq t_{k_3} < t_{k_4}$, then $z(t_{k_2}) - z(t_{k_1})$ is uncorrelated with $z(t_{k_4}) - z(t_{k_3})$ because each of these differences is made up of different ϵ's, which are themselves uncorrelated.

A Wiener process is obtained by taking the limit of the random walk process (11.9) as $\Delta t \to 0$. In symbolic form we write the equations governing a Wiener process as

$$dz = \epsilon(t)\sqrt{dt} \tag{11.11}$$

where each $\epsilon(t)$ is a standardized normal random variable. The random variables $\epsilon(t')$ and $\epsilon(t'')$ are uncorrelated whenever $t' \neq t''$.

This description of a Wiener process is not rigorous because we have no assurance that the limiting operations are defined; but it provides a good intuitive description. An alternative definition of a Wiener process can be made by simply listing the required properties. In this approach we say a process $z(t)$ is a **Wiener process** (or, alternatively, **Brownian motion**) if it satisfies the following:

1. For any $s < t$ the quantity $z(t) - z(s)$ is a normal random variable with mean zero and variance $t - s$.
2. For any $0 \leq t_1 < t_2 \leq t_3 < t_4$, the random variables $z(t_2) - z(t_1)$ and $z(t_4) - z(t_3)$ are uncorrelated.
3. $z(t_0) = 0$ with probability 1.

These properties parallel the properties of the random walk process given earlier.

It is fun to try to visualize the outcome of a Wiener process. A sketch of a possible path is shown in Figure 11.6. Remember that given $z(t)$ at time t, the value of $z(s)$ at time $s > t$ is, on average, the same as $z(t)$ but will vary from that according to a standard deviation equal to $\sqrt{s-t}$.

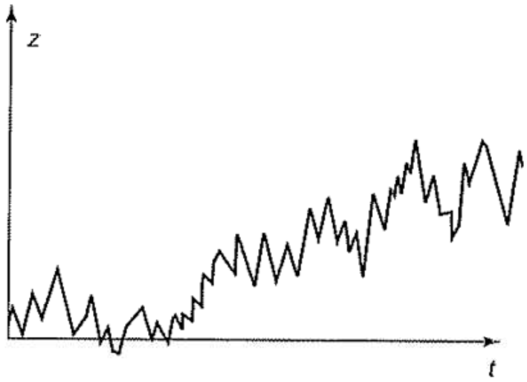

FIGURE 11.6 Path of a Wiener process. A Wiener process moves continuously but is not differentiable

A Wiener process is not differentiable with respect to time. We can roughly verify this by noting that for $t < s$,

$$E\left[\frac{z(s) - z(t)}{s - t}\right]^2 = \frac{s - t}{(s - t)^2} = \frac{1}{s - t} \to \infty$$

as $s \to t$.

It is, however, useful to have a word for the term dz/dt since this expression appears in many stochastic equations. A common word used, arising from the systems engineering field (the field that motivated Wiener's work), is **white noise.** It is really fun to try to visualize white noise. One depiction is presented in Figure 11.7.

Generalized Wiener Processes and Ito Processes

The Wiener process (or Brownian motion) is the fundamental building block for a whole collection of more general processes. These generalizations are obtained by inserting white noise in an ordinary differential equation.

The simplest extension of this kind is the **generalized Wiener process,** which is of the form

$$dx(t) = a\,dt + b\,dz \tag{11.12}$$

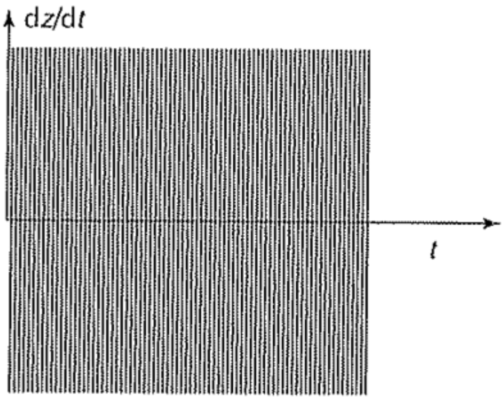

FIGURE 11.7 Fantasizing white noise. White noise is the derivative of a Wiener process, but that derivative does not exist in the normal sense

where $x(t)$ is a random variable for each t, z is a Wiener process, and a and b are constants.

A generalized Wiener process is especially important because it has an analytic solution (which can be found by integrating both sides). Specifically,

$$x(t) = x(0) + at + bz(t). \tag{11.13}$$

An **Ito process** is somewhat more general still. Such a process is described by an equation of the form

$$dx(t) = a(x, t)\,dt + b(x, t)\,dz. \tag{11.14}$$

As before, z denotes a Wiener process. Now, however, the coefficients $a(x, t)$ and $b(x, t)$ may depend on x and t, and a general solution cannot be written in an analytic form. A special form of Ito process is used frequently to describe the behavior of financial assets, as discussed in the next section.

11.7 A STOCK PRICE PROCESS

We now have the tools necessary to extend the multiplicative model of stock prices to a continuous-time model. Recall that the multiplicative model is

$$\ln S(k+1) - \ln S(k) = w(k)$$

where the $w(k)$'s are uncorrelated normal random variables. The continuous-time version of this equation is

$$d \ln S(t) = \nu\,dt + \sigma\,dz \tag{11.15}$$

where ν and $\sigma \geq 0$ are constants and z is a standard Wiener process. The whole right-hand side of the equation can be regarded as playing the role of the random variable $w(k)$ in the discrete-time model. This side can be thought of as a constant plus a normal random variable with zero mean, and hence, overall it is a normal random variable. (Although all terms in the equation are differentials or multiples of differentials and thus do not themselves have magnitude in the usual sense, it is helpful to think of dt and dz as being "small" like Δt and Δz.) The term $\nu\,dt$ is, accordingly, the mean value of the right-hand side. This mean value is proportional to dt, consistent with the fact that in the logarithm version of the multiplicative model the mean value of the change in $\ln S$ is proportional to the length of one period. The standard deviation of the right-hand side is σ times the standard deviation of dz. Hence it is of order of magnitude $\sigma\sqrt{dt}$, which is consistent with the fact that in the logarithm version of the multiplicative model the standard deviation of the change in $\ln S$ is proportional to the square root of the length of one period, as reflected by (11.7*a*) and (11.7*b*).

Since equation (11.15) is expressed in terms of $\ln S(t)$, it is actually a generalized Wiener process. Hence we can solve it explicitly using (11.13) as

$$\ln S(t) = \ln S(0) + \nu t + \sigma z(t). \tag{11.16}$$

This shows that $E[\ln S(t)] = E[\ln S(0)] + \nu t$, and hence $E[\ln S(t)]$ grows linearly with t. Because the expected logarithm of this process increases linearly with t, just as a continuously compounded bank account, this process is termed **geometric Brownian motion.**

Lognormal Prices

Like the discrete-time multiplicative model, the geometric Brownian motion process described by (11.15) is a lognormal process. This can be seen easily from the solution (11.16). The right-hand side of that equation is a normal random variable with expected value $\ln S(0) + \nu t$ and standard deviation $\sigma\sqrt{t}$.

We conclude that the price $S(t)$ itself has a lognormal distribution. We can express this formally by $\ln S(t) \sim N(\ln S(0) + \nu t, \sigma^2 t)$, where $N(m, \sigma^2)$ denotes the normal distribution with mean m and variance σ^2.

Although we can write $S(t) = \exp[\ln S(t)] = S(0)\exp[\nu t + \sigma z(t)]$, it does *not* follow that the expected value of $S(t)$ is $S(0)e^{\nu t}$. The mean value must instead be determined by equation (11.8), the general formula that applies to lognormal variables. Hence,

$$\mathrm{E}[S(t)] = S(0)e^{(\nu+\frac{1}{2}\sigma^2)t}.$$

If we define $\mu = \nu + \frac{1}{2}\sigma^2$, we have

$$\mathrm{E}[S(t)] = S(0)e^{\mu t}.$$

The standard deviation of $S(t)$ is also given by a general relation for lognormal variables. In the case of the standard deviation, the required calculation is a bit more complex. The formula is (see Exercise 5)

$$\text{stdev}[S(t)] = S(0)e^{\nu t+\frac{1}{2}\sigma^2 t}\left(e^{\sigma^2 t} - 1\right)^{1/2}.$$

Standard Ito Form

We have defined the random process for $S(t)$ in terms of $\ln S(t)$ rather than directly in terms of $S(t)$. The use of $\ln S(t)$ facilitated the development, and it highlights the fact that the process is a straightforward generalization of the multiplicative model that leads to lognormal distributions. It is, however, useful to express the process in terms of $S(t)$ itself.

In ordinary calculus we know that

$$d\ln[S(t)] = \frac{dS(t)}{S(t)}.$$

Hence we might be tempted to substitute $dS(t)/S(t)$ for $d\ln S(t)$ in the basic equation [Eq. (11.15)], obtaining $dS(t)/S(t) = \nu\, dt + \sigma\, dz$. This would be almost correct, but there is a correction term that must be applied when changing variables in Ito processes (because Wiener processes are not ordinary functions and do not follow the rules of ordinary calculus). The appropriate Ito process in terms of $S(t)$ is

$$\frac{dS(t)}{S(t)} = \left(\nu + \frac{1}{2}\sigma^2\right) dt + \sigma\, dz. \tag{11.17}$$

Note that the correction term $\frac{1}{2}\sigma^2$ is exactly the same as needed in the expression for the expected value of a lognormal random variable. Putting $\mu = \nu + \frac{1}{2}\sigma^2$, we may write the equation in the standard Ito form for price dynamics,

$$\frac{dS(t)}{S(t)} = \mu \, dt + \sigma \, dz . \tag{11.18}$$

The term $dS(t)/S(t)$ can be thought of as the differential return of the stock; hence in this form the differential return has a simple form.

The correction term required when transforming the equation from $\ln S(t)$ to $S(t)$ is a special instance of a general transformation equation defined by **Ito's lemma,** which applies to variables defined by Ito processes. Ito's lemma is discussed in the next section.

Note that if the equation in standard form is written with S in the denominator, as in (11.17), it is an equation for dS/S. This term can be interpreted as the instantaneous rate of return on the stock. Hence the standard form is often referred to as an equation for the instantaneous return.

Example 11.2 (Bond price dynamics) Let $P(t)$ denote the price of a bond that pays \$1 at time $t = T$, with no other payments. Assume that interest rates are constant at r. The price of this bond satisfies

$$\frac{dP(t)}{P(t)} = r \, dt$$

which is a deterministic Ito equation, paralleling the equation for stock prices. The solution to this equation is $P(t) = P(0)e^{rt}$. Using $P(T) = 1$, we find that $P(t) = e^{r(t-T)}$.

We now summarize the relations between $S(t)$ and $\ln S(t)$:

Relations for geometric Brownian motion *Suppose the geometric Brownian motion process $S(t)$ is governed by*

$$dS(t) = \mu S(t) \, dt + \sigma S(t) \, dz$$

where z is a standard Wiener process. Define $\nu = \mu - \frac{1}{2}\sigma^2$. Then $S(t)$ is lognormal and

$$\begin{aligned} E\{\ln[S(t)/S(0)]\} &= \nu t \\ \text{stdev}\{\ln[S(t)/S(0)]\} &= \sigma\sqrt{t} \\ E\{S(t)/S(0)\} &= e^{\mu t} \\ \text{stdev}\{S(t)/S(0)\} &= e^{\mu t}\left(e^{\sigma^2 t} - 1\right)^{1/2} \end{aligned}$$

Simulation

A continuous-time price process can be simulated by taking a series of small time periods and then stepping the process forward period by period. There are two natural ways to do this, and they are *not* exactly equivalent.

First, consider the process in standard form defined by (11.18). We take a basic period length Δt and set $S(t_0) = S_0$, a given initial price at $t = t_0$. The corresponding simulation equation is

$$S(t_{k+1}) - S(t_k) = \mu S(t_k)\Delta t + \sigma S(t_k)\epsilon(t_k)\sqrt{\Delta t}$$

where the $\epsilon(t_k)$'s are uncorrelated normal random variables of mean 0 and standard deviation 1. This leads to

$$S(t_{k+1}) = \left[1 + \mu\,\Delta t + \sigma\epsilon(t_k)\sqrt{\Delta t}\right] S(t_k) \tag{11.19}$$

which is a multiplicative model, but the random coefficient is normal rather than lognormal, so this simulation method does not produce the lognormal price distributions that are characteristic of the underlying Ito process (in either of its forms).

A second approach is to use the log (or multiplicative) form (11.15). In discrete form this is

$$\ln S(t_{k+1}) - \ln S(t_k) = \nu\,\Delta t + \sigma\epsilon(t_k)\sqrt{\Delta t}.$$

This leads to

$$S(t_{k+1}) = e^{\nu\Delta t + \sigma\epsilon(t_k)\sqrt{\Delta t}}\,S(t_k) \tag{11.20}$$

which is also a multiplicative model, but now the random coefficient *is* lognormal.

The two methods are different, but it can be shown that their differences tend to cancel in the long run. Hence in practice, either method is about as good as the other.

Example 11.3 (Simulation by two methods) Consider a stock with an initial price of \$10 and having $\nu = 15\%$ and $\sigma = 40\%$. We take the basic time interval to be 1 week ($\Delta t = 1/52$), and we simulate the stock behavior for 1 year. Both methods described in this subsection were applied using the same random ϵ's, which were generated from a normal distribution of mean 0 and standard deviation 1. Table 11.1 gives the results. The first column shows the random variables $dz = \epsilon\sqrt{\Delta t}$ for that week. The second column lists the corresponding multiplicative factors. The value P_1 is the simulated price using the standard method as represented by (11.19). The fourth column shows the appropriate exponential factors for the second method, (11.20). The value P_2 is the simulated price using that method. Note that even at the first step the results are not identical. However, overall the results are fairly close.

TABLE 11.1
Simulation of Price Dynamics

Week	dz	$\mu + \sigma$ dz	P_1	$\nu + \sigma$ dz	P_2
0			10 0000		10 0000
1	.06476	00802	10 0802	00648	10 0650
2	− 19945	− 00664	10 0132	− 00818	9 9830
3	− 83883	− 04211	9 5916	− 04365	9 5567
4	49609	.03194	9 8980	03040	9 8517
5	− 33892	−.01438	9 7557	− 01592	9 6961
6	1 39485	08180	10 5536	08026	10 5064
7	.61869	03874	10 9625	.03720	10 9046
8	.40201	02672	11 2554	02518	11 1827
9	− 71118	− 03503	10 8612	−.03656	10 7812
10	16937	01382	11 0113	01228	10 9144
11	1 19678	07081	11 7910	06927	11 6973
12	− 14408	− 00357	11 7489	− 00511	11 6377
13	80590	04913	12 3261	04759	12 2049
26	−1 23335	− 06399	13 1428	− 06553	12 9157
39	.68140	04222	17 6850	04068	17 3668
52	.69955	.04323	15 1230	.04169	14 7564

The price process is simulated by two methods. Although they differ step by step, the overall results are similar.

11.8 ITO'S LEMMA*

We saw that the two Ito equations—for $S(t)$ and for $\ln S(t)$—are different, and that the difference is not exactly what would be expected from the application of ordinary calculus to the transformation of variables from $S(t)$ to $\ln S(t)$; an additional term $\frac{1}{2}\sigma^2$ is required. This extra term arises because the random variables have order $\sqrt{dt}$, and hence their squares produce first-order, rather than second-order, effects. There is a systematic method for making such transformations in general, and this is encapsulated in Ito's lemma:

Ito's lemma *Suppose that the random process x is defined by the Ito process*

$$dx(t) = a(x, t)\,dt + b(x, t)\,dz \tag{11.21}$$

where z is a standard Wiener process. Suppose also that the process y(t) is defined by y(t) = F(x, t). Then y(t) satisfies the Ito equation

$$dy(t) = \left(\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2\right)dt + \frac{\partial F}{\partial x}b\,dz \tag{11.22}$$

where z is the same Wiener process as in Eq. (11.21).

Proof: Ordinary calculus would give a formula similar to (11.22), but without the term with $\frac{1}{2}$.

We shall sketch a rough proof of the full formula. We expand y with respect to a change Δy. In the expansion we keep terms up to first order in Δt, but since Δx is of order $\sqrt{\Delta t}$, this means that we must expand to second order in Δx. We find

$$\begin{aligned} y + \Delta y &= F(x,t) + \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\Delta x)^2 \\ &= F(x,t) + \frac{\partial F}{\partial x}(a\,\Delta t + b\,\Delta z) + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(a\,\Delta t + b\,\Delta z)^2. \end{aligned}$$

The quadratic expression in the last term must be treated in a special way. When expanded, it becomes $a^2(\Delta t)^2 + 2ab\,\Delta t\,\Delta z + b^2(\Delta z)^2$. The first two terms of this expression are of order higher than 1 in Δt, so they can be dropped. The term $b^2(\Delta z)^2$ is all that remains. However, Δz has expected value zero and variance Δt, and hence this last term is of order Δt and cannot be dropped. Indeed, it can be shown that, in the limit as Δt goes to zero, the term $(\Delta z)^2$ is nonstochastic and is equal to Δt. Substitution of this into the previous expansion leads to

$$y + \Delta y = F(x,t) + \left(\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2\right)\Delta t + \frac{\partial F}{\partial x}b\,\Delta z.$$

Taking the limit and using $y = F(x,t)$ yields Ito's equation, (11.22). ∎

Example 11.4 (Stock dynamics) Suppose that $S(t)$ is governed by the geometric Brownian motion

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}z.$$

Let us use Ito's lemma to find the equation governing the process $F(S(t)) = \ln S(t)$.

We have the identifications $a = \mu S$ and $b = \sigma S$. We also have $\partial F/\partial S = 1/S$ and $\partial^2 F/\partial S^2 = -1/S^2$. Therefore according to (11.22),

$$\begin{aligned} \mathrm{d}\ln S &= \left(\frac{a}{S} - \frac{1}{2}\frac{b^2}{S^2}\right)\mathrm{d}t + \frac{b}{S}\mathrm{d}z \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\,\mathrm{d}z \end{aligned}$$

which agrees with our earlier result.

11.9 BINOMIAL LATTICE REVISITED

Let us consider again the binomial lattice model shown in Figure 11.8 (which is identical to Figure 11.1). The model is analogous to the multiplicative model discussed earlier in this chapter, since at each step the price is multiplied by a random variable

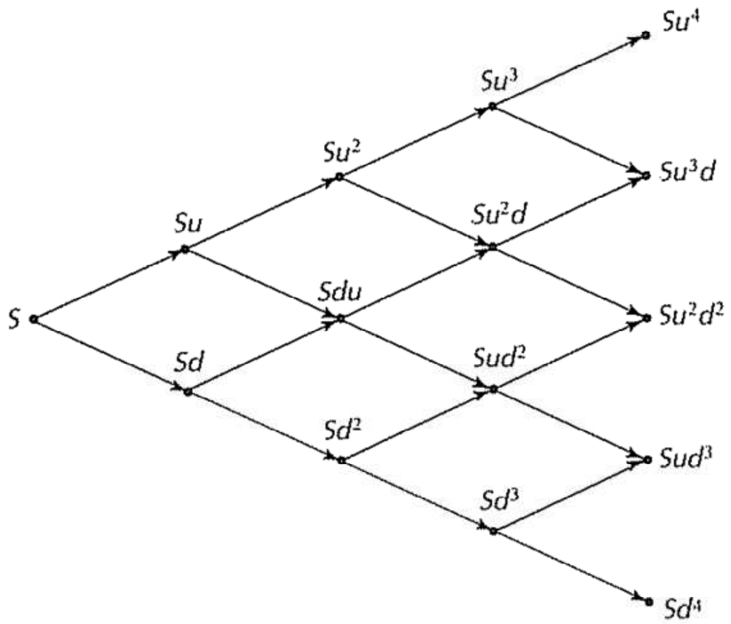

FIGURE 11.8 Binomial lattice stock model. At each step the stock price S either increases to uS or decreases to dS

In this case, the random variable takes only the two possible values u and d. We can find suitable values for u, d, and p by matching the multiplicative model as closely as possible. This is done by matching both the expected value of the logarithm of a price change and the variance of the logarithm of the price change.[6]

To carry out the matching, it is only necessary to ensure that the random variable S_1, which is the price after the first step, has the correct properties since the process is identical thereafter. Taking $S(0) = 1$, we find by direct calculation that

$$\begin{aligned} \mathrm{E}(\ln S_1) &= p \ln u + (1-p) \ln d \\ \mathrm{var}(\ln S_1) &= p(\ln u)^2 + (1-p)(\ln d)^2 - [p \ln u + (1-p) \ln d]^2 \\ &= p(1-p)(\ln u - \ln d)^2 . \end{aligned}$$

Therefore the appropriate parameter matching equations are

$$pU + (1-p)D = \nu \, \Delta t \tag{11.23}$$

$$p(1-p)(U-D)^2 = \sigma^2 \Delta t \tag{11.24}$$

where $U = \ln u$ and $D = \ln d$.

Notice that three parameters are to be chosen: U, D, and p; but there are only two requirements. Therefore there is one degree of freedom. One way to use this freedom is to set $D = -U$ (which is equivalent to setting $d = 1/u$). In this case the

[6]For the lattice, the probability of attaining the various end nodes of the lattice is given by the binomial distribution. Specifically, the probability of reaching the value $Su^k d^{n-k}$ is $\binom{n}{k} p^k (1-p)^{n-k}$, where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ is the binomial coefficient. This distribution approaches (in a certain sense) a normal distribution for large n. The logarithm of the final prices is of the form $k \ln u + (n-k) \ln d$, which is linear in k. Hence the distribution of the end point prices can be considered to be nearly lognormal

equations (11.23) and (11.24) reduce to

$$(2p-1)U = \nu\,\Delta t$$
$$4p(1-p)U^2 = \sigma^2\Delta t\,.$$

If we square the first equation and add it to the second, we obtain

$$U^2 = \sigma^2\,\Delta t + (\nu\,\Delta t)^2\,.$$

Substituting this in the first equation, we may solve for p directly, and then $U = \ln u$ can be determined. The resulting solutions to the parameter matching equations are

$$\begin{aligned} p &= \frac{1}{2} + \frac{\frac{1}{2}}{\sqrt{\sigma^2/(\nu^2\Delta t)+1}} \\ \ln u &= \sqrt{\sigma^2\Delta t + (\nu\,\Delta t)^2} \\ \ln d &= -\sqrt{\sigma^2\Delta t + (\nu\,\Delta t)^2}\,. \end{aligned} \tag{11.25}$$

For small Δt (11.25) can be approximated as

$$\begin{aligned} p &= \tfrac{1}{2} + \tfrac{1}{2}\left(\frac{\nu}{\sigma}\right)\sqrt{\Delta t} \\ u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}} \end{aligned} \tag{11.26}$$

These are the values presented in Section 11.1.

11.10 SUMMARY

A simple and versatile model of asset dynamics is the binomial lattice. In this model an asset's price is assumed to be multiplied either by the factor u or by the factor d, the choice being made each period according to probabilities p and $1-p$, respectively. This model is used extensively in theoretical developments and as a basis for computing solutions to investment problems.

Another broad class of models are those where the asset price may take on values from a continuum of possibilities. The simplest model of this type is the additive model. If the random inputs of this model are normal random variables, the asset prices are also normal random variables. This model has the disadvantage, however, that prices may be negative.

A better model is the multiplicative model of the form $S(k+1) = u(k)S(k)$. If the multiplicative inputs $u(k)$ are lognormal, then the future prices $S(k)$ are also lognormal. The model can be expressed in the alternative form as $\ln S(k+1) - \ln S(k) = \ln u(k)$.

By letting the period length tend to zero, the multiplicative model becomes the Ito process $\mathrm{d}\ln S(t) = \nu\,\mathrm{d}t + \sigma^2\mathrm{d}z(t)$, where z is a normalized Wiener process. This special form of an Ito process is called geometric Brownian motion. This model can be expressed in the alternative (but equivalent) form $\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma^2 S(t)\mathrm{d}z(t)$, where $\mu = \nu + \frac{1}{2}\sigma^2$.

Ito processes are useful representations of asset dynamics. An important tool for transforming such processes is Ito's lemma: If $x(t)$ satisfies an Ito process, and $y(t)$ is defined by $y(t) = F(x, t)$, Ito's lemma specifies the process satisfied by $y(t)$.

A binomial lattice model can be considered to be an approximation to an Ito process. The parameters of the lattice can be chosen so that the mean and standard deviation of the logarithm of the return agree in the two models.

EXERCISES

1. (Stock lattice) A stock with current value $S(0) = 100$ has an expected growth rate of its logarithm of $\nu = 12\%$ and a volatility of that growth rate of $\sigma = 20\%$. Find suitable parameters of a binomial lattice representing this stock with a basic elementary period of 3 months. Draw the lattice and enter the node values for 1 year. What are the probabilities of attaining the various final nodes?

2. (Time scaling) A stock price S is governed by the model

$$\ln S(k+1) = \ln S(k) + w(k)$$

where the period length is 1 month. Let $\nu = \mathrm{E}[w(k)]$ and $\sigma^2 = \mathrm{var}[w(k)]$ for all k. Now suppose the basic period length is changed to 1 year. Then the model is

$$\ln S(K+1) = \ln S(K) + W(K)$$

where each movement in K corresponds to 1 year. What is the natural definition of $W(K)$? Show that $\mathrm{E}[W(K)] = 12\nu$ and $\mathrm{var}[W(K)] = 12\sigma^2$. Hence parameters scale in proportion to time.

3. (Arithmetic and geometric means) Suppose that $v_1, v_2, \ldots, v_n$ are positive numbers. The *arithmetic mean* and the *geometric mean* of these numbers are, respectively,

$$v_A = \frac{1}{n}\sum_{i=1}^{n} v_i \qquad \text{and} \qquad v_G = \left(\prod_{i=1}^{n} v_i\right)^{1/n}$$

(*a*) It is always true that $v_A \geq v_G$. Prove this inequality for $n = 2$.

(*b*) If $r_1, r_2, \ldots, r_n$ are rates of return of a stock in each of n periods, the arithmetic and geometric mean rates of return are likewise

$$r_A = \frac{1}{n}\sum_{i=1}^{n} r_i \qquad \text{and} \qquad r_G = \left(\prod_{i=1}^{n}(1+r_i)\right)^{1/n} - 1$$

Suppose \$40 is invested. During the first year it increases to \$60 and during the second year it decreases to \$48. What are the arithmetic and geometric mean rates of return over the 2 years?

(*c*) When is it appropriate to use these means to describe investment performance?

4. (Complete the square ◇) Suppose that $u = e^w$, where w is normal with expected value $\overline{w}$ and variance σ^2. Then

$$\overline{u} = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{w} e^{-(w-\overline{w})^2/2\sigma^2}\, dw.$$

Show that

$$w - \frac{(w - \overline{w})^2}{2\sigma^2} = -\frac{1}{2\sigma^2}\left[w - (\overline{w} + \sigma^2)\right]^2 + \overline{w} + \frac{\sigma^2}{2}$$

Use the fact that

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-(x-\bar{x}^2)/2\sigma^2}\,dx = 1$$

to evaluate $\overline{u}$.

5. (Log variance ◇) Use the method of Exercise 4 to find the variance of a lognormal variable in terms of the parameters of the underlying normal variable

6. (Expectations) A stock price is governed by geometric Brownian motion with $\mu = .20$ and $\sigma = .40$ The initial price is $S(0) = 1$ Evaluate the four quantities

$$\mathrm{E}[\ln S(1)], \qquad \mathrm{stdev}[\ln S(1)]$$
$$\mathrm{E}[S(1)], \qquad \mathrm{stdev}[S(1)].$$

7. (Application of Ito's lemma) A stock price S is governed by

$$dS = aS\,dt + bS\,dz$$

where z is a standardized Wiener process Find the process that governs

$$G(t) = S^{1/2}(t).$$

8. (Reverse check) Gavin Jones was mystified by Ito's lemma when he first studied it, so he tested it He started with S governed by

$$dS = \mu S\,dt + \sigma S\,dz$$

and found that $Q = \ln S$ satisfies

$$dQ = (\mu - \tfrac{1}{2}\sigma^2)\,dt + \sigma\,dz.$$

He then applied Ito's lemma to this last equation using the change of variable $S = e^Q$. Duplicate his calculations What did he get?

9. (Two simulations ◇) A useful expansion is

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \cdots$$

Use this to express the exponential in equation (11 20) in linear terms of powers of Δt up to first order. Note that this differs from the expression in (11.19), so conclude that the standard form and the multiplicative (or lognormal) form of simulation are different even to first order Show, however, that the expected values of the two expressions *are* identical to first order, and hence, over the long run the two methods should produce similar results

10. (A simulation experiment ⊕) Consider a stock price S governed by the geometric Brownian motion process

$$\frac{dS}{S(t)} = .10\,dt + .30\,dz$$

(*a*) Using $\Delta t = 1/12$ and $S(0) = 1$, simulate several (i.e., *many*) years of this process using either method, and evaluate

$$\frac{1}{t}\ln[S(t)]$$

as a function of t. Note that it tends to a limit p. What is the theoretical value of this limit?

(*b*) About how large must t be to obtain two-place accuracy?

(*c*) Evaluate

$$\frac{1}{t}\left[\ln S(t) - pt\right]^2$$

as a function of t Does this tend to a limit? If so, what is its theoretical value?

REFERENCES

For a good overview of stock models similar to this chapter, see [1] For greater detail on stochastic processes see [2], and for general information of how stock prices actually behave, see [3].

There are numerous textbooks on probability theory that discuss the normal distribution and the lognormal distribution A classic is [4] The book by Wiener [5] was responsible for inspiring a great deal of serious theoretical and practical work on issues involving Wiener processes Ito's lemma was first published in [6] and later in [7].

1 Hull, J. C (1993), *Options, Futures, and Other Derivative Securities*, 2nd ed , Prentice Hall, Englewood Cliffs, NJ
2 Malliaris, A G , and W A Brock (1982), *Stochastic methods in Economics and Finance*, North-Holland, Amsterdam, The Netherlands.
3 Cootner, P H , Ed (1964), *The Random Character of Stock Market Prices*, M I T Press, Cambridge, MA
4. Feller, W (1950), *Probability Theory and Its Applications*, vols 1 and 2, Wiley, New York.
5 Wiener, N (1950), *Extrapolation, Interpolation, and Smoothing of Stationary Time Series*, Technology Press, M I T , Cambridge, MA, and Wiley, New York.
6. Ito, K (1951), "On a Formula Concerning Stochastic Differentials," *Nagoya Mathematics Journal*, **3,** 55–65
7 Ito, K. (1961), *Lectures on Stochastic Processes*, Tata Institute of Fundamental Research, India.