Asset Pricing

John H. Cochrane

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Chapter 6. Relation between discount factors, betas, and mean-variance frontiers

In this chapter, I draw the connection between discount factors, mean-variance frontiers, and beta representations. In the first chapter, I showed how mean-variance and a beta representation follow from $p = E(mx)$ and (in the mean-variance case) complete markets. Here, I discuss the connections in both directions and in incomplete markets, drawing on the representations studied in the last chapter.

The central theme of the chapter is that *all three representations are equivalent.* Figure 18 summarizes the ways one can go from one representation to another. A discount factor, a reference variable for betas – the thing you put on the right hand side in the regressions that give betas – and a return on the mean-variance frontier all carry the same information, and given any one of them, you can find the others. More specifically,

1. $p = E(mx) \Rightarrow \beta$: Given m such that $p = E(mx)$, $m,\ x^*$, R^*, or $R^* + wR^{e*}$ all can serve as reference variables for betas.
2. $p = E(mx) \Rightarrow$ mean-variance frontier. You can construct R^* from $x^* = proj(m|\underline{X})$, $R^* = x^*/E(x^{*2})$, and then R^*, $R^* + wR^{e*}$ are on the mean-variance frontier.
3. Mean-variance frontier $\Rightarrow p = E(mx)$. If R^{mv} is on the mean-variance frontier, then $m = a + bR^{mv}$ linear in that return is a discount factor; it satisfies $p = E(mx)$.
4. $\beta \Rightarrow p = E(mx)$. If we have an expected return/beta model with factors f, then $m = b'f$ linear in the factors satisfies $p = E(mx)$ (and vice-versa).
5. If a return is on the mean-variance frontier, then there is an expected return/beta model with that return as reference variable.

The following subsections discuss the mechanics of going from one representation to the other in detail. The last section of the chapter collects some special cases when there is no riskfree rate. The next chapter discusses some of the implications of these equivalence theorems, and why they are important.

Roll (197x) pointed out the connection between mean-variance frontiers and beta pricing. Ross (1978) and Dybvig and Ingersoll (1982) pointed out the connection between linear discount factors and beta pricing. Hansen and Richard (1987) pointed out the connection between a discount factor and the mean-variance frontier.

6.1 From discount factors to beta representations

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m, x^*, and R^* can all be the single factor in a single beta representation.

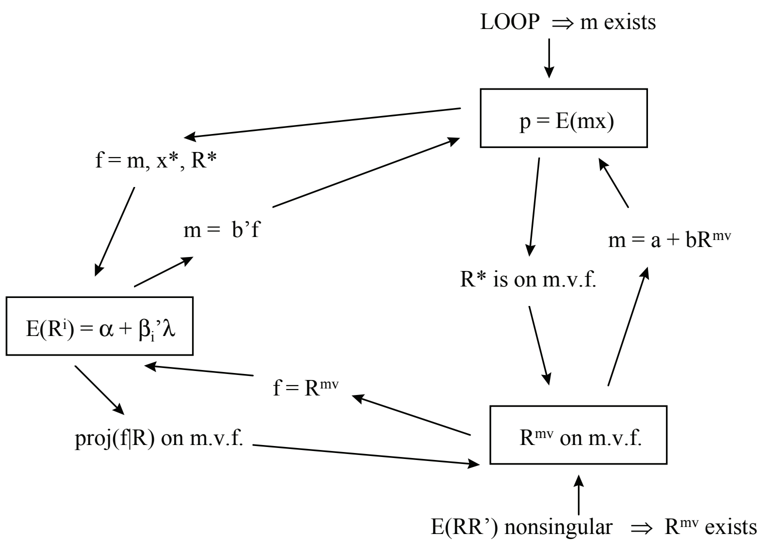

Figure 18. Relation between three views of asset pricing.

6.1.1 Beta representation using m

$p = E(mx)$ implies $E(R^i) = \alpha + \beta_{i,m}\lambda_m$. Start with

$$1 = E(mR^i) = E(m)E(R^i) + cov(m, R^i).$$

Thus,

$$E(R^i) = \frac{1}{E(m)} - \frac{cov(m, R^i)}{E(m)}.$$

Multiply and divide by $var(m)$, define $\alpha \equiv 1/E(m)$ to get

$$E(R^i) = \alpha + \left(\frac{cov(m, R^i)}{var(m)}\right)\left(-\frac{var(m)}{E(m)}\right) = \alpha + \beta_{i,m}\lambda_m.$$

As advertised, we have a single beta representation.

For example, we can equivalently state the consumption-based model as: mean asset returns should be linear in the regression betas of asset returns on $(c_{t+1}/c_t)^{-\gamma}$. Furthermore, the slope of this cross-sectional relationship λ_m is not a free parameter, though it is usually treated as such in empirical evaluation of factor pricing models. λ_m should equal the ratio of variance to mean of $(c_{t+1}/c_t)^{-\gamma}$.

The factor risk premium λ_m for marginal utility growth is negative. Positive expected returns are associated with positive correlation with consumption growth, and hence negative correlation with marginal utility growth and m. Thus, we expect $\lambda_m < 0$.

6.1.2 β representation using x^* and R^*

It is often useful to express a pricing model in a way that the factor is a payoff rather than a real factor such as consumption growth. In applications, we can then avoid measurement difficulties of real data. We have already seen the idea of "factor mimicking portfolios" formed by projection: project m on to $\underline{X}$, and the resulting payoff x^* also serves as a discount factor. Unsurprisingly, x^* can also serve as the factor in an expected return-beta representation. It's even more useful if the reference payoff is a return. Unsurprisingly, the return $R^* = x^*/E(x^{*2})$ can also serve as the factor in a beta pricing model. When the factor is also a return, the model is particularly simple, since the factor risk premium is also the expected excess return.

Theorem. $1 = E(mR^i)$ implies an expected return - beta model with $x^* = proj(m|\underline{X})$ or $R^* \equiv x^*/E(x^{*2})$ as factors, e.g. $E(R^i) = \alpha + \beta_{i,x^*}\lambda_{x^*}$ and $E(R^i) = \alpha + \beta_{i,R^*}[E(R^*) - \alpha]$.

Proof: Recall that $p = E(mx)$ implies $p = E[proj(m \mid \underline{X})\, x]$, or $p = E(x^* x)$. Then

$$1 = E(mR^i) = E(x^* R^i) = E(x^*)E(R^i) + cov(x^*, R^i).$$

Solving for the expected return,

$$E(R^i) = \frac{1}{E(x^*)} - \frac{cov(x^*, R^i)}{E(x^*)} = \frac{1}{E(x^*)} - \frac{cov(x^*, R^i)}{var(x^*)}\frac{var(x^*)}{E(x^*)} \tag{81}$$

which we can write as the desired single-beta model,

$$E(R^i) = \alpha + \beta_{i,x^*}\lambda_{x^*}.$$

Notice that the zero-beta rate $1/E(x^*)$ appears when there is no riskfree rate.

To derive a single beta representation with R^*, recall the definition,

$$R^* = \frac{x^*}{E(x^{*2})}$$

Substituting R^* for x^*, equation (6.81) implies that we can in fact construct a return R^* from m that acts as the single factor in a beta model,

$$E(R^i) = \frac{E(R^{*2})}{E(R^*)} - \frac{cov(R^*, R^i)}{E(R^*)} = \frac{E(R^{*2})}{E(R^*)} + \left(\frac{cov(R^*, R^i)}{var(R^*)}\right)\left(-\frac{var(R^*)}{E(R^*)}\right)$$

or, defining Greek letters in the obvious way,

$$E(R^i) = \alpha + \beta_{R^i,R^*}\lambda_{R^*} \tag{82}$$

Since the factor R^* is also a return, its expected excess return over the zero beta rate gives the factor risk premium λ_{R^*}. Applying equation (6.82) to R^* itself,

$$E(R^*) = \alpha - \frac{var(R^*)}{E(R^*)}. \tag{83}$$

So we can write the beta model in an even more traditional form

$$E(R^i) = \alpha + \beta_{R^i,R^*}[E(R^*) - \alpha]. \tag{84}$$

■

Recall that R^* is the minimum second moment frontier, on the lower portion of the mean-variance frontier. This is why R^* has an unusual negative expected excess return or factor risk premium, $\lambda_{R^*} = -var(R^*)/E(R^*) < 0$. α is the zero-beta rate on R^* shown in Figure15.

Special cases

A footnote to these constructions is that $E(m)$, $E(x^*)$, or $E(R^*)$ cannot be zero, or you couldn't divide by them. This is a pathological case: $E(m) = 0$ implies a zero price for the riskfree asset, and an infinite riskfree rate. If a riskfree rate is traded, we can simply observe that it is not infinite and verify the fact. Also, in a complete market, $E(m)$ cannot be zero since, by absence of arbitrage, $m > 0$. We will see similar special cases in the remaining theorems: the manipulations only work for discount factor choices that do not imply zero or infinite riskfree rates. I discuss the issue in section 6.6

The manipulation from expected return-covariance to expected return-beta breaks down if $var(m)$, $var(x^*)$ or $var(R^*)$ are zero. This is the case of pure risk neutrality. In this case, the covariances go to zero faster than the variances, so all betas go to zero and all expected returns become the same as the risk free rate.

6.2 From mean-variance frontier to a discount factor and beta representation

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R^{mv} is on mean-variance frontier $\Rightarrow m = a + bR^{mv};\ E(R^i) - \alpha = \beta_i\,[E(R^{mv}) - \alpha]$

We have seen that $p = E(mx)$ implies a single$-\beta$ model with *a* mean-variance efficient reference return, namely R^*. The converse is also true: for (almost) any return on the mean-variance frontier, we can define a discount factor m that prices assets as a linear function of the mean-variance efficient return. Also, expected returns mechanically follow a single$-\beta$ representation using the mean-variance efficient return as reference.

I start with the discount factor.

Theorem: There is a discount factor of the form $m = a + bR^{mv}$ if and only if R^{mv} is on the mean-variance frontier, and $R^{mv}$is not the riskfree rate. (If there is no riskfree rate, if R^{mv} is not the constant-mimicking portfolio return.)

Graphical argument

The basic idea is very simple, and Figure 19 shows the geometry for the complete markets case. The discount factor $m = x^*$ is proportional to R^*. The mean-variance frontier is $R^* + wR^{e*}$. Pick a vector R^{mv} on the mean-variance frontier as shown in Figure 19. Then stretch it (bR^{mv}) and then subtract some of the 1 vector (a). Since R^{e*} is generated by the unit vector, we can get rid of the R^{e*} component and get back to the discount factor x^* if we pick the right a and b.

If the original return vector were not on the mean-variance frontier, then any linear combination $a + bR^{mv}$ with $b \neq 0$ would point in some of the n direction, which R^* and x^* do not. If $b = 0$, though, just stretching up and down the 1 vector will not get us to x^*. Thus, we can *only* get a discount factor of the form $a + bR^{mv}$ if R^{mv} is on the frontier.

You may remember that x^* is not the only discount factor – all discount factors are of the form $m = x^* + \varepsilon$ with $E(\varepsilon x) = 0$. Perhaps $a + bR$ gives one of these discount factors, when R is not on the mean-variance frontier? This doesn't work, however; n is still in the payoff space $\underline{X}$ while, by definition, ε is orthogonal to this space.

If the mean-variance efficient return R^{mv} that we start with happens to lie right on the intersection of the stretched unit vector and the frontier, then stretching the R^{mv} vector and adding some unit vector are the same thing, so we again can't get back to x^* by stretching and adding some unit vector. The stretched unit payoff is the riskfree rate, so the theorem rules out the riskfree rate. When there is no riskfree rate, we have to rule out the "constant-mimicking portfolio return." I treat this case in section 6.1.

Algebraic proof

Now, an algebraic proof that captures the same ideas.

Proof. For an arbitrary R, try the discount factor model

$$m = a + bR = a + b(R^* + wR^{e*} + n). \tag{85}$$

We show that this discount factor prices an arbitrary payoff if and only if $n = 0$, and except for the w choice that makes R the riskfree rate, or the constant-mimicking portfolio return if there is no riskfree rate.

We can determine a and b by forcing m to price any two assets. I find a and b to

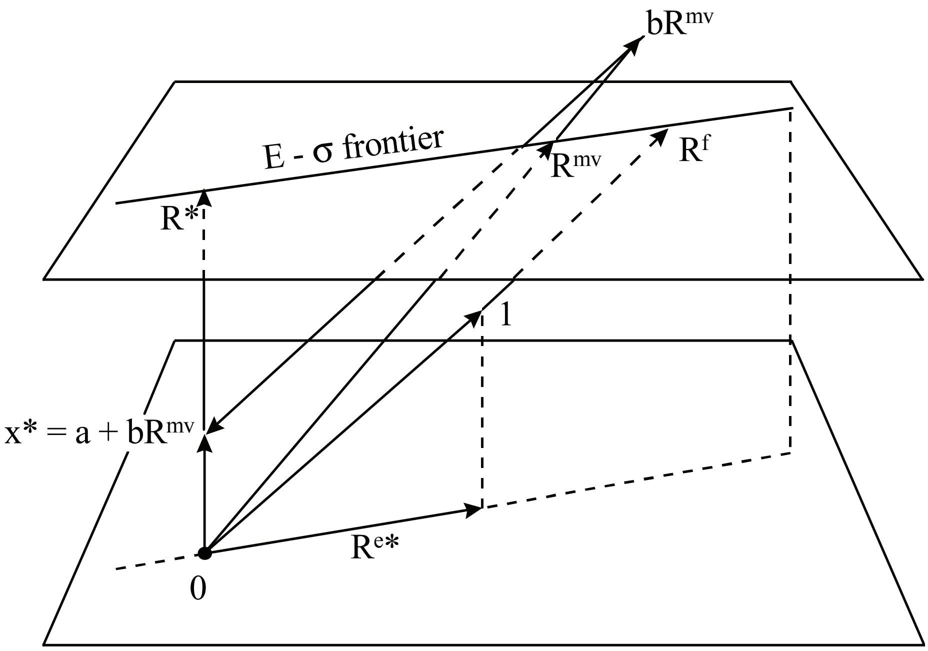

Figure 19. There is a discount factor $m = a + bR^{mv}$ if and only if R^{mv} is on the mean-variance frontier and not the risk free rate.

make the model price R^* and R^{e*}.

$$\begin{aligned} 1 &= E(mR^*) = aE(R^*) + bE(R^{*2}) \\ 0 &= E(mR^{e*}) = aE(R^{e*}) + bwE(R^{e*2}) = (a + bw)\, E(R^{e*}). \end{aligned}$$

Solving for a and b,

$$\begin{aligned} a &= \frac{w}{wE(R^*) - E(R^{*2})} \\ b &= -\frac{1}{wE(R^*) - E(R^{*2})}. \end{aligned}$$

Thus, if it is to price R^* and R^{e*}, the discount factor must be

$$m = \frac{w - (R^* + wR^{e*} + n)}{wE(R^*) - E(R^{*2})}. \tag{86}$$

Now, let's see if m prices an arbitrary payoff x^i. Any $x^i \in \underline{X}$ can also be decomposed as

$$x^i = y^i R^* + w^i R^{e*} + n^i.$$

(See Figure 14 if this isn't obvious.) The price of x^i is y^i, since both R^{e*} and n^i are zero-price (excess return) payoffs. Therefore, we want $E(mx^i) = y^i$. Does it?

$$E(mx^i) = E\left(\frac{(w - R^* - wR^{e*} - n)(y^i R^* + w^i R^{e*} + n^i)}{wE(R^*) - E(R^{*2})}\right)$$

Using the orthogonality of R^*, R^{e*} n; $E(n) = 0$ and $E(R^{e*2}) = E(R^{e*})$ to simplify the product,

$$E(mx^i) = \frac{wy^i E(R^*) - y^i E(R^{*2}) - E(nn^i)}{wE(R^*) - E(R^{*2})} = y^i - \frac{E(nn^i)}{wE(R^*) - E(R^{*2})}.$$

To get $p(x^i) = y^i = E(mx^i)$, we need $E(nn^i) = 0$. The only way to guarantee this condition for *every* payoff $x^i \in \underline{X}$ is to insist that $n = 0$.
Obviously, this construction can't work if the denominator of (6.86) is zero, i.e. if $w = E(R^{*2})/E(R^*) = 1/E(x^*)$. If there is a riskfree rate, then $R^f = 1/E(x^*)$, so we are ruling out the case $R^{mv} = R^* + R^f R^{e*}$, which is the risk free rate. If there is no riskfree rate, I interpret $\hat{R} = R^* + E(R^{*2})/E(R^*)R^{e*}$ as a "constant mimicking portfolio return" in section 5.3, and I give a graphical interpretation of this special case in section 6.1 ■

We can generalize the theorem somewhat. Nothing is special about returns; any payoff of the form $yR^* + wR^{e*}$ or $yx^* + wR^{e*}$ can be used to price assets; such payoffs have minimum variance among all payoffs with given mean and price. Of course, we proved existence not uniqueness: $m = a + bR^{mv} + \epsilon,\ E(\epsilon x) = 0$ also price assets as always.

To get from the mean-variance frontier to a beta pricing model, we can just chain this theorem and the theorem of the last section together. There is a slight subtlety about special cases when there is no riskfree rate, but since it is not important for the basic points I relegate the direct connection and the special cases to section 6.2.

6.3 Factor models and discount factors

Beta-pricing models are equivalent to linear models for the discount factor m.

$$E(R^i) = \alpha + \lambda' \beta_i \Leftrightarrow m = a + b' f$$

We have shown that $p = E(mx)$ implies a single beta representation using $m,\ x^*$ or R^* as factors. Let's ask the converse question: suppose we have an expected return - beta model such as CAPM, APT, ICAPM, etc. What discount factor model does this imply? I show that an expected return - beta model is equivalent to a model for the discount factor that is a linear function of the factors in the beta model. This is an important and central result. It gives the connection between the discount factor formulation emphasized in this book and the expected return/beta, factor model formulation common in empirical work.

You can write a linear factor model most compactly as $m = b'f$, letting one of the factors be a constant. However, since we want a connection to the beta representation based on covariances rather than second moments, it is easiest to fold means of the factors in to the constant, and write $m = a + b'f$ with $E(f) = 0$ and hence $E(m) = a$.

The connection is easiest to see in the special case that all the test assets are excess returns. Then $0 = E(mR^e)$ does not identify the mean of m, and we can normalize a arbitrarily. I find it convenient to normalize to $E(m) = 1$, or $m = 1 + b'\left[f - E\left(f\right)\right]$. Then,

Theorem: Given the model

$$m = 1 + b'\left[f - E\left(f\right)\right];\ 0 = E(mR^e) \tag{87}$$

one can find λ such that

$$E(R^e) = \beta'\lambda \tag{88}$$

where β are the multiple regression coefficients of excess returns R^e on the factors.
Conversely, given λ in (6.88), we can find b such that (6.87) holds.
Proof: From (6.87)

$$\begin{aligned} 0 &= E(mR^e) = E(R^e) + b'cov(f, R^e) \\ E(R^e) &= -b'cov(f, R^e). \end{aligned}$$

From covariance to beta is quick,

$$E(R^e) = -b'var(f)var(f)^{-1}cov(f, R^e) = \lambda'\beta$$

Thus, λ and b are related by

$$\lambda = -var(f)b.$$

■

When the test assets are returns, the same idea works just as well, but gets a little more drowned in algebra since we have to keep track of the constant in m and the zero-beta rate in the beta model.

Theorem: Given the model

$$m = a + b'f, \ 1 = E(mR^i), \tag{89}$$

one can find α and λ such that

$$E(R^i) = \alpha + \lambda'\beta_i, \tag{90}$$

where β_i are the multiple regression coefficients of R^i on f with a constant. Conversely, given α and λ in a factor model of the form (6.90), one can find $a, \ b$ such that (6.89) holds.
Proof: We just have to construct the relation between (α, λ) and (a, b) and show that it works. Start with $m = a + b'f$, $1 = E(mR)$, and hence

$$E(R) = \frac{1}{E(m)} - \frac{cov(m, R)}{E(m)} = \frac{1}{a} - \frac{E(Rf')b}{a} \tag{91}$$

β_i is the vector of the appropriate regression coefficients,

$$\beta_i \equiv E\left(ff'\right)^{-1} E(fR^i),$$

so to get β in the formula, continue with

$$E(R) = \frac{1}{a} - \frac{E(Rf')E(ff')^{-1}E(ff')b}{a} = \frac{1}{a} - \beta' \frac{E(ff')b}{a}$$

Now, define α and λ to make it work,

$$\begin{aligned} \alpha &\equiv \frac{1}{E(m)} = \frac{1}{a} \\ \lambda &\equiv -\frac{1}{a}E(ff')b = -\alpha E[mf] \end{aligned} \tag{6.92}$$

Using (6.92) we can just as easily go backwards from the expected return-beta representation to $m = a + b'f$.
As always, we have to worry about a special case of zero or infinite riskfree rates. We rule out $E(m) = E(a + b'f) = 0$ to keep (6.91) from exploding, and we rule out $\alpha = 0$ and $E(ff')$ singular to go from α, β, λ in (6.92) back to m. ■

Given either model *there is* a model of the other form. They are not *unique*. We can add to m any random variable orthogonal to returns, and we can add spurious risk factors with zero β and/or λ , leaving pricing implications unchanged. We can also express the multiple beta model as a single beta model with $m = a + b'f$ as the single factor, or use its corresponding R^*.

Equation (6.92) shows that the factor risk premium λ can be interpreted as the price of the factor; A test of $\lambda \neq 0$ is often called a test of whether the "factor is priced." More precisely,

λ captures the price $E(mf)$ of the (de-meaned) factors brought forward at the risk free rate. If we start with underlying factors $\tilde{f}$ such that the demeaned factors are $f = \tilde{f} - E(\tilde{f})$,

$$\lambda \equiv -\alpha \, p \left[\tilde{f} - E(\tilde{f}) \right] = -\alpha \left[p(\tilde{f}) - \frac{E(\tilde{f})}{\alpha} \right]$$

λ represents the price of the factors less their risk-neutral valuation, i.e. the *factor risk premium*. If the factors are not traded, λ is the model's predicted price rather than a market price. Low prices are high risk premia, resulting in the negative sign. If the factors are returns with price one, then the factor risk premium is the expected return of the factor, less α, $\lambda = E(f) - \alpha$.

Note that the "factors" need not be returns (though they may be); they need not be orthogonal, and they need not be serially uncorrelated or conditionally or unconditionally mean-zero. Such properties may occur as natural special cases, or as part of the economic derivation of specific factor models, but they are not required for the existence of a factor pricing representation. For example, if the riskfree rate is constant then $E_t(m_{t+1})$ is constant and at least the sum $b'f$ should be uncorrelated over time. But if the riskfree rate is not constant, then $E_t(m_{t+1}) = E_t(b'f_{t+1})$ *should* vary over time.

Factor-mimicking portfolios

It is often convenient to use factor-mimicking payoffs

$$f^* = proj(f|\underline{X})$$

factor-mimicking returns

$$f^* = \frac{proj(f|\underline{X})}{p\left[proj(f|\underline{X})\right]}$$

or factor-mimicking excess returns

$$f^* = proj(f|\underline{R^e})$$

in place of true factors. These payoffs carry the same pricing information as the original factors, and can serve as reference variables in expected return-beta representations

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When the factors are not already returns or excess returns, it is convenient to express a beta pricing model in terms of its *factor mimicking portfolios* rather than the factors themselves. Recall that $x^* = proj(m|\underline{X})$ carries all of $m's$ pricing implications on $\underline{X}$; $p(x) = E(mx) = E(x^*x)$. The factor-mimicking portfolios are just the same idea using the individual factors.

Define the payoffs f^* by

$$f^* = proj(f|\underline{X})$$

Then, $m = b'f^*$carries the same pricing implications on $\underline{X}$ as does $m = b'f$:

$$p = E(mx) = E(b'f\ x) = E\left[b'(projf|\underline{X})\ x\right] = E\left[b'f^*\ x\right]. \tag{93}$$

(I include the constant as one of the factors.)

The factor-mimicking portfolios also form a beta representation. Just go from (6.93) back to an expected return- beta representation

$$E(R^i) = \alpha^* + \beta^{*\prime}\lambda^*, \tag{94}$$

and find λ^*, α^* using (6.92). The β^* are the regression coefficients of the returns R^i on the factor-mimicking portfolios, not on the factors, as they should be.

It is more traditional to use the *returns* or *excess returns* on the factor-mimicking portfolios rather than payoffs as I have done so far. To generate returns, divide the payoff by its price,

$$f^* = \frac{proj(f|\underline{X})}{p\left[proj(f|\underline{X})\right]}.$$

The resulting b will be scaled down by the price of the factor-mimicking payoff, and the model is the same. Note you project on the space of payoffs, not of returns. Returns $\underline{R}$ are not a space, since they don't contain zero.

If the test assets are all excess returns, you can even more easily project the factors on the set of excess returns, which are a space since they do include zero. If we define

$$f^* = proj(f|\underline{R^e})$$

then of course the excess returns f^* carry the same pricing implications as the factors f for a set of excess returns; $m = b'f^*$ satisfies $0 = E(mR^{ei})$ and

$$E(R^{ei}) = \beta_{i,f^*}\lambda = \beta_{i,f^*}E(f^*)$$

6.4 Discount factors and beta models to mean - variance frontier

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From m, we can construct R^* which is on the mean variance frontier

If a beta pricing model holds, then the return R^* on the mean-variance frontier is a linear combination of the factor-mimicking portfolio returns.

Any frontier return is a combination of R^* and one other return, a risk free rate or a risk free rate proxy. Thus, any frontier return is a linear function of the factor-mimicking returns plus a risk free rate proxy.

It's easy to show that given m that we can find *a* return on the mean-variance frontier. Given m construct $x^* = proj(m|\underline{X})$ and $R^* = x^*/E(x^{*2})$. R^* is the minimum second moment return, and hence on the mean-variance frontier.

Similarly, if you have a set of factors f for a beta model, then a linear combination of the factor-mimicking portfolios is on the mean-variance frontier. A beta model is the same as $m = b'f$. Since m is linear in f, x^* is linear in $f^* = proj(f|\underline{X})$, so R^* is linear in the factor mimicking payoffs f^* or their returns $f^*/p(f^*)$.

Section 5.4 showed how we can span the mean-variance frontier with R^* and a risk free rate, if there is one, or the zero-beta, minimum variance, or constant-mimicking portfolio return $\hat{R} = proj(1|\underline{X})/p[proj(1|\underline{X})]$ if there is no risk free rate. The latter is particularly nice in the case of a linear factor model, since we may consider the constant as a factor, so the frontier is entirely generated by factor-mimicking portfolio returns.

6.5 Three riskfree rate analogues

I introduce three counterparts to the risk free rate that show up in asset pricing formulas when there is no risk free rate. The three returns are the *zero-beta* return, the *minimum-variance return* and the *constant-mimicking portfolio* return.

Three different generalizations of the riskfree rate are useful when a risk free rate or unit payoff is not in the set of payoffs. These are the *zero-beta* return, the *minimum-variance return* and the *constant-mimicking portfolio* return. I introduce the returns in this section, and I use them in the next section to state some special cases involving the mean-variance frontier. Each of these returns maintains one property of the risk free rate in a market in which there is no risk free rate. The zero-beta return is a mean-variance efficient return that is uncorrelated with another given mean-variance efficient return. The minimum-variance return is just that. The constant-mimicking portfolio return is the return on the payoff "closest" to the unit payoff. Each of these returns one has a representation in the standard form $R^* + wR^{e*}$ with slightly different w. In addition, the expected returns of these risky assets are used in some asset pricing representations. For example, the zero beta *rate* is often used to refer to the expected value of the zero beta return.

Each of these riskfree rate analogues is mean-variance efficient. Thus, I characterize each one by finding its weight w in a representation of the form $R^* + wR^{e*}$. We derived such a representation above for the riskfree rate as equation (5.72),

$$R^f = R^* + R^f R^{e*}. \tag{95}$$

In the last subsection, I show how each riskfree rate analogue reduces to the riskfree rate when there is one.

6.5.1 Zero-beta return for R^*

The zero beta return for R^*, denoted R^α, is the mean-variance efficient return uncorrelated with R^*. Its expected return is the zero beta rate $\alpha = E(R^a)$. This zero beta return has representation

$$R^a = R^* + \frac{var(R^*)}{E(R^*)E(R^{e*})} R^{e*},$$

and the corresponding zero beta rate is

$$\alpha = E(R^\alpha) = \frac{E(R^{*2})}{E(R^*)} = \frac{1}{E(x^*)}.$$

The zero beta rate is found graphically in mean-standard deviation space by extending the tangency at R^* to the vertical axis. It is also the inverse of the price that x^* and R^* assign to the unit payoff.

The riskfree rate R^f is of course uncorrelated with R^*. Risky returns uncorrelated with R^* earn the same average return as the risk free rate if there is one, so they might take the place of R^f when the latter does not exist. For any return R^α that is uncorrelated with R^* we have $E(R^* R^\alpha) = E(R^*)E(R^\alpha)$, so

$$\alpha = E(R^\alpha) = \frac{E(R^{*2})}{E(R^*)} = \frac{1}{E(x^*)}.$$

The first equality introduces a popular notation α for this rate. I call α the zero beta *rate*, and R^a the zero beta *return*. There is no riskfree rate, so there is no security that just pays α.

As you can see from the formula, the zero-beta rate is the inverse of the price that R^* and x^* assign to the unit payoff, which is another natural generalization of the riskfree rate. It is called the zero *beta* rate because $cov(R^*, R^\alpha) = 0$ implies that the regression beta of R^α on R^* is zero. More precisely, one might call it the zero beta rate *on* R^*, since one can calculate zero-beta rates for returns other than R^* and they are not the same as the zero-beta rate for R^* In particular, the zero-beta rate on the "market portfolio" will generally be different from the zero beta rate on R^*.

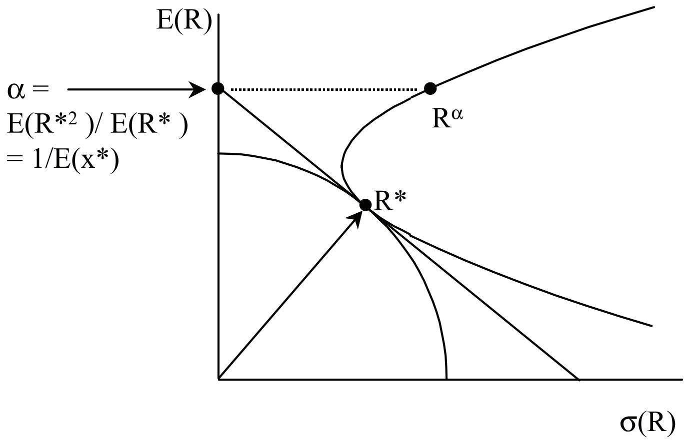

Figure 20. Zero-beta rate α and zero-beta return R^{α} for R^{*}.

I draw α in Figure 20 as the intersection of the tangency and the vertical axis. This is a property of any return on the mean variance frontier: The expected return on an asset uncorrelated with the mean-variance efficient asset (a *zero-beta* asset) lies at the point so constructed. To check this geometry, use similar triangles: The length of R^* in Figure 20 is $\sqrt{E(R^{*2})}$, and its vertical extent is $E(R^*)$. Therefore, $\alpha/\sqrt{E(R^{*2})} = \sqrt{E(R^{*2})}/E(R^*)$, or $\alpha = E(R^{*2})/E(R^*)$. Since R^* is on the lower portion of the mean-variance frontier, this zero beta rate α is above the minimum variance return.

Note that in general $\alpha \neq 1/E(m)$. Projecting m on $\underline{X}$ preserves asset pricing implications on $\underline{X}$ but not for payoffs not in $\underline{X}$. Thus if a risk free rate is not traded, x^* and m may differ in their predictions for the riskfree rate as for other nontraded assets.

The zero beta *return* is the rate of return on the mean-variance frontier with mean equal to the zero beta rate, as shown in Figure 20. We want to characterize this return in $R^* + wR^{e*}$ form. To do this, we want to find w such that

$$E(R^{\alpha}) = \frac{E(R^{*2})}{E(R^*)} = E(R^*) + wE(R^{e*}).$$

Solving, the answer is

$$w = \frac{E(R^{*2}) - E(R^*)^2}{E(R^*)E(R^{e*})} = \frac{var(R^*)}{E(R^*)E(R^{e*})}.$$

Thus, the zero beta *return* is

$$R^{\alpha} = R^* + \frac{var(R^*)}{E(R^*)E(R^{e*})}R^{e*},$$

expression (6.103). Note that the weight is not $E(R^{\alpha}) = E(R^{*2})/E(R^*)$. When there is no risk free rate, the weight and the mean return are different.

6.5.2 Minimum variance return

The minimum variance return has the representation

$$R^{\text{min. var.}} = R^* + \frac{E(R^*)}{1 - E(R^{e*})}R^{e*}.$$

The riskfree rate obviously is the minimum variance return when it exists. When there is no risk free rate, the minimum variance return is

$$R^{\text{min. var.}} = R^* + \frac{E(R^*)}{1 - E(R^{e*})}R^{e*}. \tag{96}$$

Taking expectations,

$$E(R^{\text{min. var.}}) = E(R^*) + \frac{E(R^*)}{1 - E(R^{e*})} E(R^{e*}) = \frac{E(R^*)}{1 - E(R^{e*})}.$$

The minimum variance return retains the nice property of the risk free rate, that its weight on R^{e*} is the same as its mean,

$$R^{\text{min. var.}} = R^* + E(R^{\text{min. var.}}) R^{e*}$$

just as $R^f = R^* + R^f R^{e*}$. When there is no risk free rate, the zero-beta and minimum variance returns are not the same. You can see this fact clearly in Figure 20.

We can derive expression (6.96) for the minimum variance return by brute force: choose w in $R^* + wR^{e*}$ to minimize variance.

$$\min_w \; var(R^* + wR^{e*}) = E[(R^* + wR^{e*})^2] - E(R^* + wR^{e*})^2 =$$

$$= E(R^{*2}) + w^2 E(R^{e*}) - E(R^*)^2 - 2wE(R^*)E(R^{e*}) - w^2 E(R^{e*})^2.$$

The first order condition is

$$0 = wE(R^{e*})[1 - E(R^{e*})] - E(R^*)E(R^{e*})$$

$$w = \frac{E(R^*)}{1 - E(R^{e*})}.$$

6.5.3 Constant-mimicking portfolio return

The constant-mimicking portfolio return is defined as the return on the projection of the unit vector on the payoff space,

$$\hat{R} = \frac{proj(1|\underline{X})}{p\,[proj(1|\underline{X})]}.$$

It has the representation

$$\hat{R} = R^* + \frac{E(R^{*2})}{E(R^*)} R^{e*}.$$

When there is a risk free rate, it is the rate of return on a unit payoff, $R^f = 1/p(1)$. When there is no risk free rate, we might define the rate of return on the mimicking portfolio for a unit payoff,

$$\hat{R} = \frac{proj(1|\underline{X})}{p\left[proj(1|\underline{X})\right]}.$$

I call this object the *constant-mimicking portfolio return.*

The mean-variance representation of the constant-mimicking portfolio return is

$$\hat{R} = R^* + \alpha R^{e*} = R^* + \frac{E(R^{*2})}{E(R^*)} R^{e*}. \tag{97}$$

Note that the weight α equal to the zero beta rate creates the constant-mimicking return, not the zero beta return. To show (6.97), start with property (5.73),

$$R^{e*} = proj(1|\underline{X}) - \frac{E(R^*)}{E(R^{*2})} R^*. \tag{98}$$

Take the price of both sides. Since the price of R^{e*} is zero and the price of R^* is one, we establish

$$p\left[proj(1|\underline{X})\right] = \frac{E(R^*)}{E(R^{*2})}. \tag{99a}$$

Solving (6.98) for $proj(1|\underline{X})$, dividing by (6.99a) we obtain the right hand side of (6.97).

6.5.4 Risk free rate

The risk free rate has the mean-variance representation

$$R^f = R^* + R^f R^{e*}.$$

The zero-beta, minimum variance and constant-mimicking portfolio returns reduce to this formula when there is a risk free rate.

Again, we derived in equation (5.72) that the riskfree rate has the representation,

$$R^f = R^* + R^f R^{e*}. \tag{100}$$

Obviously, we should expect that the zero-beta return, minimum-variance return, and constant-mimicking portfolio return reduce to the riskfree rate when there is one. These other rates are

$$\text{constant-mimicking: } \hat{R} = R^* + \frac{E(R^{*2})}{E(R^*)} R^{e*} \tag{101}$$

$$\text{minimum-variance: } R^{\text{min. var.}} = R^* + \frac{E(R^*)}{1 - E(R^{e*})} R^{e*} \tag{102}$$

$$\text{zero-beta: } R^{\alpha} = R^* + \frac{var(R^*)}{E(R^*)E(R^{e*})} R^{e*}. \tag{103}$$

To establish that these are all the same when there is a riskfree rate, we need to show that

$$R^f = \frac{E(R^{*2})}{E(R^*)} = \frac{E(R^*)}{1 - E(R^{e*})} = \frac{var(R^*)}{E(R^*)E(R^{e*})} \tag{104}$$

We derived the first equality above as equation (5.74). To derive the second equality, take expectations of (6.95),

$$R^f = E(R^*) + R^f E(R^{e*}) \tag{105}$$

and solve for R^f. To derive the third equality, use the first equality from (6.104) in (6.105),

$$\frac{E(R^{*2})}{E(R^*)} = E(R^*) + R^f E(R^{e*}).$$

Solving for R^f,

$$R^f = \frac{E(R^{*2}) - E(R^*)^2}{E(R^*)E(R^{e*})} = \frac{var(R^*)}{E(R^*)E(R^{e*})}.$$

6.6 Mean-variance special cases with no riskfree rate

We can find a discount factor from any mean-variance efficient return except the constant-mimicking return.

We can find a beta representation from any mean-variance efficient return except the minimum-variance return.

I collect in this section the special cases for the equivalence theorems of this chapter. The special cases all revolve around the problem that the expected discount factor, price of a unit payoff or riskfree rate must not be zero or infinity. This is typically an issue of theoretical rather than practical importance. In a complete, arbitrage free market, $m > 0$ so we know $E(m) > 0$. If a riskfree rate is traded you can observe $\infty > E(m) = 1/R^f > 0$. However, in an incomplete market in which no riskfree rate is traded, there are many discount factors with the same asset pricing implications, and you might have happened to choose one with $E(m) = 0$ in your manipulations. By and large, this is easy to avoid: choose another of the many discount factors with the same pricing implications that does not have $E(m) = 0$. More generally, when you choose a particular discount factor you are choosing an extension of the current set of prices and payoffs; you are viewing the current prices and payoffs as a subset of a particular contingent-claim economy. Make sure you pick a sensible one. Therefore, we could simply state the special cases as "when a riskfree rate is not traded, make sure you use discount factors with $0 < E(m) < \infty$." However, it is potentially useful and it certainly is traditional to specify the special *return* on the mean-variance frontier that leads to the infinite or zero implied riskfree rate, and to rule it out directly. This section works out what those returns are and shows why they must be avoided.

6.6.1 The special case for mean variance frontier to discount factor

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When there is no riskfree rate, we can find a discount factor that is a linear function of any mean-variance efficient return except the constant-mimicking portfolio return.

In section 6.2, we saw that we can form a discount factor $a + bR^{mv}$ from any mean-variance efficient return R^{mv} except one particular return, of the form $R^* + \frac{E(R^{*2})}{E(R^*)} R^{e*}$. This return led to an infinite m. We now recognize this return as the risk-free rate, when there is one, or the constant-mimicking portfolio return, if there is no riskfree rate.

Figure 21 shows the geometry of this case. To use no more than three dimensions I had to reduce the return and excess return spaces to lines. The payoff space $\underline{X}$ is the plane joining the return and excess return sets as shown. The set of all discount factors is $m = x^* + \varepsilon$, $E(\varepsilon x) = 0$, the line through x^* orthogonal to the payoff space $\underline{X}$ in the figure. I draw the unit payoff (the dot marked "1" in Figure 21) closer to the viewer than the plane $\underline{X}$, and I draw a vector through the unit payoff coming out of the page.

Take any return on the mean-variance frontier, R^{mv}. (Since the return space only has two dimensions, all returns are on the frontier.) For a given R^{mv}, the space $a + bR^{mv}$ is the plane spanned by R^{mv} and the unit payoff. This plane lies sideways in the figure. As the figure shows, there is a vector $a + bR^{mv}$ in this plane that lies on the line of discount factors.

Next, the special case. This construction would go awry if the plane spanning the unit payoff and the return R^{mv} were parallel to the plane containing the discount factor. Thus, the construction would not work for the return marked $\hat{R}$ in the Figure. This is a return corresponding to a payoff that is the projection of the unit payoff on to $\underline{X}$, so that the residual will be orthogonal to $\underline{X}$, as is the line of discount factors.

With Figure 21 in front of us, we can also see why the constant-mimicking portfolio return is not the same thing as the minimum-variance return. Variance is the size or second moment

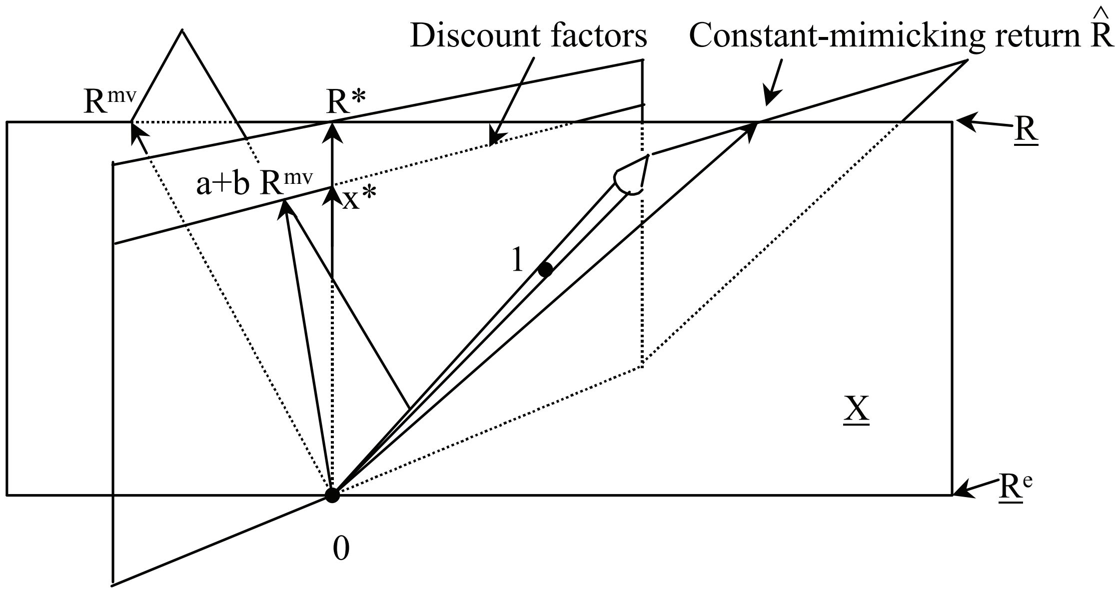

Figure 21. One can construct a discount factor $m = a + bR^{mv}$ from any mean-variance-efficient return except the constant-mimicking return $\hat{R}$.

of the residual in a projection (regression) on 1.

$$var(x) = E\left[(x - E(x))^2\right] = E\left[(x - proj(x|1))^2\right] = ||x - proj(x|1)||^2$$

Thus, the minimum variance return is the return closest to extensions of the unit vector. It is formed by projecting *returns* on the *unit vector*. The constant-mimicking portfolio return is the return on the payoff closest to 1 It is formed by projecting the *unit vector* on the set of *payoffs*.

6.6.2 The special case for mean-variance frontier to a beta model

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We can use any return on the mean-variance frontier as the reference return for a single beta representation, except the minimum-variance return.

We already know mean variance frontiers ⇔ discount factor and discount factor ⇔ single beta representation, so at a superficial level we can string the two theorems together to go from a mean-variance efficient return to a beta representation. However it is more elegant to go directly, and the special cases are also a bit simpler this way.

Theorem: There is a single beta representation with a return R^{mv} as factor,

$$E(R^i) = \alpha_{R^{mv}} + \beta_{i,R^{mv}} \left[E(R^{mv}) - \alpha \right],$$

if and only if R^{mv} is mean-variance efficient and not the minimum variance return.

This famous theorem is given by Roll (1976) and Hansen and Richard (1987). We rule out minimum variance to rule out the special case $E(m) = 0$. Graphically, the zero-beta rate is formed from the tangency to the mean-variance frontier as in Figure 20. I use the notation $\alpha_{R^{mv}}$ to emphasize that we use the zero-beta rate corresponding to the particular mean-variance return R^{mv} that we use as the reference return. If we used the minimum-variance return, that would lead to an infinite zero-beta rate.

Proof: The mean-variance frontier is $R^{mv} = R^* + wR^{e*}$. Any return is $R^i = R^* + w^i R^{e*} + n^i$. Thus,

$$E(R^i) = E(R^*) + w^i E(R^{e*}) \tag{106}$$

Now,

$$\begin{aligned} cov(R^i, R^{mv}) &= cov\left[(R^* + wR^{e*}), (R^* + w^i R^{e*}) \right] \\ &= var(R^*) + ww^i var(R^{e*}) - (w + w^i) E(R^*) E(R^{e*}) \\ &= var(R^*) - wE(R^*)E(R^{e*}) + w^i \left[w\ var(R^{e*}) - E(R^*)E(R^{e*}) \right] \end{aligned}$$

Thus, $cov(R^i, R^{mv})$ and $E(R^i)$ are both linear functions of w^i. We can solve $cov(R^i, R^{mv})$ for w^i, plug into the expression for $E(R^i)$ and we're done.
To do this, of course, we must be able to solve $cov(R^i, R^{mv})$ for w^i. This requires

$$w \neq \frac{E(R^*)E(R^{e*})}{var(R^{e*})} = \frac{E(R^*)E(R^{e*})}{E(R^{e*2}) - E(R^{e*})^2} = \frac{E(R^*)}{1 - E(R^{e*})} \tag{107}$$

which is the condition for the minimum variance return. ■

6.7 Problems

1. In the argument that R^{mv} on the mean variance frontier, $R^{mv} = R^* + wR^{e*}$, implies a discount factor $m = a + bR^{mv}$, do we have to rule out the case of risk neutrality? (Hint: What is R^{e*} when the economy is risk-neutral?)
2. If you use factor mimicking portfolios as in (6.93), you know that the predictions for expected returns are the same as they are if you use the factors themselves . Are the α^*,

λ^*, and β^* for the factor mimicking portfolio representation the same as the original α, λ, and β of the factor pricing model?

3. Suppose the CAPM is true, $m = a - bR^m$ prices a set of assets, and there is a risk-free rate R^f. Find R^* in terms of the moments of R^m, R^f.
4. If you express the mean-variance frontier as a linear combination of factor-mimicking portfolios from a factor model, do the relative weights of the various factor portfolios in the mean-variance efficient return change as you sweep out the frontier, or do they stay the same? (Start with the riskfree rate case)
5. For an arbitrary mean-variance efficient return of the form $R^* + wR^{e*}$, find its zero-beta return and zero-beta rate. Show that your rate reduces to the riskfree rate when there is one.
6. When the economy is risk neutral, and if there is no risk-free rate, show that the zero-beta, minimum-variance, and constant-mimicking portfolio returns are again all equivalent, though not equal to the risk-free rate. (In this case, the mean-variance frontier is just the minimum-variance point.)