Auxiliar extra C1

Considere el campo $\mathbf{F}(\mathbf{r}) = \hat{r} + g(r)\,\hat{\theta}$ en coordenadas esféricas, donde $r = \sqrt{\mathbf{r}\cdot\mathbf{r}}$ es la magnitud del vector posición y $g : \mathbb{R} \to \mathbb{R}$ es un campo escalar de clase $\mathcal{C}^1$. Entonces, $\nabla \cdot \mathbf{F}$ es:

- ○ 0
- ○ $1/r$
- ○ $2/r$
- ○ $g(r)/r$

$$\vec{F} = 1\,\hat{r} + g\,\hat{\theta} + 0\,\hat{\varphi}$$

$$\operatorname{div} \vec{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(F_u h_v h_w) + \frac{\partial}{\partial v}(h_u F_v h_w) + \frac{\partial}{\partial w}(h_u h_v F_w)\right]$$

(r, θ, φ)

$$\boxed{\begin{aligned} h_r &= 1 \\ h_\theta &= r\,\text{sen}\,\varphi \end{aligned}}$$

$h_\varphi = r$

$F_r = 1$

$F_\theta = g$

$\boxed{F_\varphi = 0}$

$$\nabla \cdot \vec{F} = \frac{1}{r^2 \text{sen}\,\varphi}\left[\partial_r(1 \cdot r^2 \text{sen}\,\varphi) + \cancel{\partial_\theta\left[r\,g(r)\right]}^{\,0} + \cancel{\partial_\varphi(r\,\text{sen}\,\varphi \cdot 0)}^{\,0}\right]$$

$$= \frac{1}{r^2 \cancel{\text{sen}\,\varphi}}\cancel{\text{sen}\,\varphi}\;\partial_r(r^2) = \frac{2r}{r^2} = \frac{2}{r}$$

Considere el sistema de coordenadas dado por $\mathbf{r}(x, \rho, \theta) = x\,\hat{x} + \rho\,\cos\theta\,\hat{y} + \rho\,\sin\theta\,\hat{y}$, con $x \in \mathbb{R}$, $\rho > 0$ y $\theta \in [0, 2\pi]$. El operador ∇ se expresa en la base $\mathcal{B} : \{\hat{x}, \hat{\rho}, \hat{\theta}\}$ como:

- ● $\hat{x}\,\frac{\partial}{\partial x} + \hat{\rho}\,\frac{\partial}{\partial \rho} + \frac{1}{\rho}\,\hat{\theta}\,\frac{\partial}{\partial \theta}$ ✓ •
- ○ $\hat{x}\,\frac{\partial}{\partial x} + \cancel{\frac{1}{\rho}\,\hat{\rho}\,\frac{\partial}{\partial \rho}} + \hat{\theta}\,\frac{\partial}{\partial \theta}$ ✗
- ○ $\cancel{\frac{1}{x}\,\hat{x}\,\frac{\partial}{\partial x}} + \hat{\rho}\,\frac{\partial}{\partial \rho} + \frac{1}{\rho}\,\hat{\theta}\,\frac{\partial}{\partial \theta}$ ✗
- ○ $\hat{x}\,\frac{\partial}{\partial x} + \hat{\rho}\,\frac{\partial}{\partial \rho} + \cancel{\hat{\theta}\,\frac{\partial}{\partial \theta}}$ ✗

$$\left[\frac{d}{dx}\right] = \frac{1}{L} \qquad [\nabla] = \frac{1}{L}$$

$$[\nabla] = \left[\hat{x}\left[\frac{\partial}{\partial x}\right] + \hat{y}\left(\frac{\partial}{\partial y}\right) + \hat{z}\right.$$

$$\frac{1}{L} \qquad \frac{1}{L} \qquad \frac{1}{L}$$

Auxiliar extra C1

En aguas subterráneas, la ecuación de Darcy $\mu k^{-1}\mathbf{u}(x,z) = -\nabla p(x,z) - \rho(x,z)\, g\, \hat{z}$ modela el escurrimiento de fluidos bajo el suelo, donde μ, k y g son constantes físicas del modelo. Si la vorticidad se define como $\omega = \nabla \times \mathbf{u}$, es válido que:

- ○ $\omega = \cancel{\left(\frac{\partial p}{\partial x} + \frac{kg}{\mu}\frac{\partial \rho}{\partial y}\right)\hat{x}}$
- ● $\omega = \frac{kg}{\mu}\frac{\partial \rho}{\partial x}\hat{y}$
- ○ $\omega = \cancel{\left(\frac{\partial p}{\partial y} + \frac{kg}{\mu}\frac{\partial \rho}{\partial z}\right)\hat{y}}$
- ○ $\omega = \cancel{\frac{kg}{\mu}\frac{\partial \rho}{\partial x}\hat{z}}$

$$\vec{f}(x) = \vec{g}(y)\, h(z) \;/\; \nabla\times$$

$$\vec{u} = \frac{-k}{\mu}\left[\nabla p + \rho g \hat{z}\right] \;/\; \nabla\times$$

$$\nabla\times(\nabla\phi) = 0$$

$$\nabla\times\vec{u} = \frac{-k}{\mu}\Big[\cancelto{0}{\nabla\times(\nabla p)} + \nabla\times(\underbrace{\rho g \hat{z}}_{\vec{F}})\Big]$$

$$=$$

$$F_x = 0$$
$$F_y = 0$$
$$F_z = \rho g$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \rho g \end{vmatrix}$$

$$\nabla\times\vec{F} = \hat{x}\left[\partial_z F_y - \partial_y F_z\right] + \hat{y}\left[\partial_z F_x - \partial_x F_z\right] + \hat{z}\left[\partial_x F_y - \partial_y F_x\right]$$

Auxiliar extra C1

$$\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \hat{x} \cdot 0 + \hat{y} \cdot 0 + \hat{z} \cdot \left(\partial_x \left(\frac{x}{x^2+y^2} \right) - \partial_y \left(\frac{-y}{x^2+y^2} \right) \right)$$

$$= \hat{z} \left[\partial_x \left(\underline{x} (x^2+y^2)^{-1} \right) + \partial_y \left(\underline{y} (x^2+y^2)^{-1} \right) \right]$$

$$= \hat{z} \left[1 (x^2+y^2)^{-1} + x(-1)(x^2+y^2)^{-2} 2x + \underline{1 (x^2+y^2)^{-1}} + y(-1)(x^2+y^2)^{-2} 2y \right]$$

$$= \hat{z} \left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} \right]$$

$$= \hat{z} \left[\frac{2}{x^2+y^2} - \frac{2x^2+2y^2}{(x^2+y^2)^2} \right] = \hat{z} \, \frac{\cancel{2(x^2+y^2)} - \cancel{(2x^2+2y^2)}}{(x^2+y^2)^2} = 0$$

$\vec{0}$ $\qquad \mathbb{R}^3 - \{(0,0)\}$

$$\text{Dom}(\vec{F}) = \mathbb{R}^3 \setminus \{(x,y) = \vec{0}\}$$

$$\text{Dom}(\nabla \times \vec{F}) = \mathbb{R}^3 \setminus \{(0,0)\}$$

Auxiliar extra C1

$$\int_\gamma \vec{F}\cdot d\vec{r} \,, \quad \gamma : \begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \qquad \theta \in [0, 2\pi)$$

$$\vec{F} = \left(\frac{-y}{x^2+y^2} , \frac{x}{x^2+y^2} , 0 \right)$$

$$\vec{F}(\gamma) = \left(\frac{-\sin\theta}{\cos^2\theta + \sin^2\theta} , \frac{\cos\theta}{\cos^2\theta+\sin^2\theta} , 0 \right)$$

$$\vec{F}(\gamma) = \left(-\sin\theta , \cos\theta , 0 \right)$$

$$d\vec{r} = \left(dx , dy , dz \right)$$

$$= \left(-\sin\theta\, d\theta , \cos\theta\, d\theta , 0 \right)$$

$$dx = d\cos\theta$$
$$= -\sin\theta\, d\theta$$
$$dy = d\sin\theta$$
$$= \cos\theta\, d\theta$$
$$dz = 0$$

$$\vec{F}\cdot d\vec{r} = \sin^2 d\theta + \cos^2\theta\, d\theta$$

$$\oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \sin^2\theta\, d\theta + \cos^2\theta\, d\theta$$

$$= \int_0^{2\pi} \left(\cancelto{1}{\sin^2\theta + \cos^2\theta} \right) d\theta$$

$$= \int_0^{2\pi} 1\, d\theta = 2\pi$$

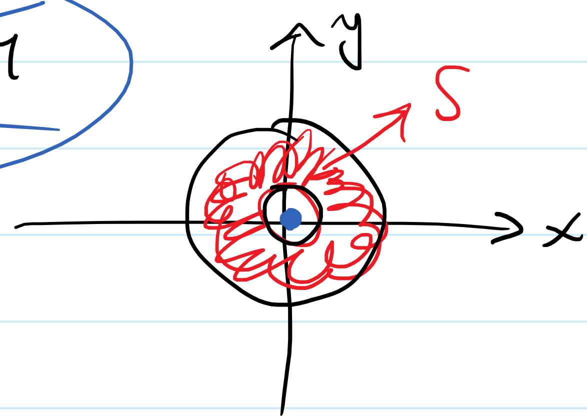
Auxiliar extra C1

Si $\boxed{\text{- - - - - - - - -}}$

$$\cancel{\underbrace{\iint_S (\nabla \times \vec{F})\, d\vec{S}}_{0} = \underbrace{\oint_{\partial} \vec{F} \cdot d\vec{r}}_{2\pi}}$$

$$\cancel{0 \neq 2\pi}$$

$$\mathrm{Dom}(\vec{F}) = \mathbb{R}^3 - \{(0,0)\}$$



$$\frac{x}{x^2+y^2}$$

$$\vec{F} = \left(\frac{1}{x^2 - 2y + 1},\ \log(y),\ \frac{1}{e^x} \right)$$

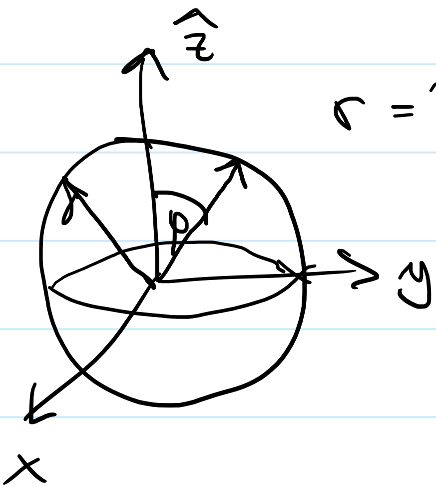
Auxiliar extra C1

$\vec{F}$ es conservativo

$\Rightarrow \exists \phi, \quad \vec{F} = \nabla \phi$

$\Rightarrow \displaystyle\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}\cdot d\vec{r} = \phi(\vec{r}_f) - \phi(\vec{r}_i) \Leftrightarrow \oint \vec{F}\cdot d\vec{r} = 0$

$x^2 + y^2 + z^2 = R^2$



$r = R$

$x = r \operatorname{sen}\varphi \cos\theta$

$y = r \operatorname{sen}\varphi \operatorname{sen}\theta$

$\boxed{z = r\cos\varphi}$

$x = R\operatorname{sen}\varphi\cos\theta \qquad \varphi \in [0,\pi]$

$y = R\operatorname{sen}\varphi\operatorname{sen}\theta \qquad \theta \in [0, 2\pi)$

$z = R\cos\varphi$

$$\vec{F} = \frac{R^3\cos^3\varphi\,\hat{z}}{3}$$

$$d\vec{S} = \hat{n}\, dS = \hat{r}\, h_\varphi h_\theta\, d\varphi\, d\theta = \hat{r} R^2 \operatorname{sen}\varphi\, d\varphi\, d\theta$$

Auxiliar extra C1

$$\vec{F}\cdot d\vec{S} = \frac{R^3\cos^3\varphi}{3}\, R^2 \sin\varphi\, d\varphi\, d\theta\, (\underbrace{\hat{z}\cdot\hat{r}}_{\cos\varphi})$$

$$= \frac{R^5}{3}\cos^4\varphi \sin\varphi\, d\varphi\, d\theta$$

$$\frac{R^5}{3}\int_0^{2\pi}\int_0^{\pi} \cos^4\varphi \sin\varphi\, \underline{d\varphi}\, d\theta = \frac{R^5}{3}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\cos^4\varphi\sin\varphi\, d\varphi\right)$$

$$= \frac{2\pi R^5}{3}\int_0^{\pi}\cos^4\varphi \sin\varphi\, d\varphi$$

$\varphi \to 0 \Rightarrow u \to 1$

$\varphi \to \pi \Rightarrow u \to -1$

$u = \cos\varphi$

$du = -\sin\varphi\, d\varphi$

$$= -\frac{2\pi R^5}{3}\int_1^{-1} u^4\, du$$

$$= -\frac{2\pi R^5}{3}\left.\frac{u^5}{5}\right|_{-1}^{1}$$

$$= -\frac{2\pi R^5}{3}\left(\frac{(-1)^5}{5} - \frac{(1)^5}{5}\right)$$

$$= -\frac{2\pi R^5}{3}\left(\frac{-1}{5} - \frac{1}{5}\right)$$

$$= \frac{4\pi R^5}{15}$$

Auxiliar extra C1

$$\iiint_V \nabla \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$$

$$\vec{F} = \frac{z^3}{3}\hat{z} \qquad \nabla \cdot \vec{F} = z^2$$

$$= r^2 \cos^2 \varphi$$

$$dV = r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \cos^2\varphi \, r^2 \sin\varphi \; dr\, d\varphi\, d\theta$$

$$u = \cos\varphi \rightarrow du = -\sin\varphi\, d\varphi$$

$$= \left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^{\pi} \cos^2\varphi \sin\varphi\, d\varphi\right)\left(\int_0^R r^4\, dr\right)$$

$\varphi \rightarrow 0 \Rightarrow u = 1$

$\varphi \rightarrow \pi \Rightarrow u = -1$

$$= (2\pi)\left(-\int_1^{-1} u^2\, d\varphi\right) \cdot \frac{r^5}{5}\Big|_0^R \qquad -\left((-1)^3 - 1\right)$$

$$-(-1-1) = -\,-2$$

$$= 2\pi \frac{R^5}{5}\left(-\frac{u^3}{3}\Big|_1^{-1}\right) = \frac{4\pi R^5}{15}$$

Auxiliar extra C1

$\hat{r} = (\operatorname{sen}\varphi\cos\theta, \operatorname{sen}\varphi\operatorname{sen}\theta, \cos\varphi)$ $\qquad \hat{z} = \cos\varphi\,\hat{r} - \operatorname{sen}\varphi\,\hat{\theta}$

$$\hat{r}\cdot\hat{z} = \left(\operatorname{sen}\varphi\cos\theta\,\underline{\hat{x}} + \operatorname{sen}\varphi\operatorname{sen}\theta\,\hat{y} + \boxed{\cos\varphi\,\hat{z}}\right)\cdot\hat{z}$$

$$= \cos\varphi\,\hat{z}\cdot\hat{z}$$