a) $(1-i)^4(1+i)^4$

$= \left((1-i)(1+i)\right)^4$

$= (1-i^2)^4$

$= (1-(-1))^4 = 16+0i$

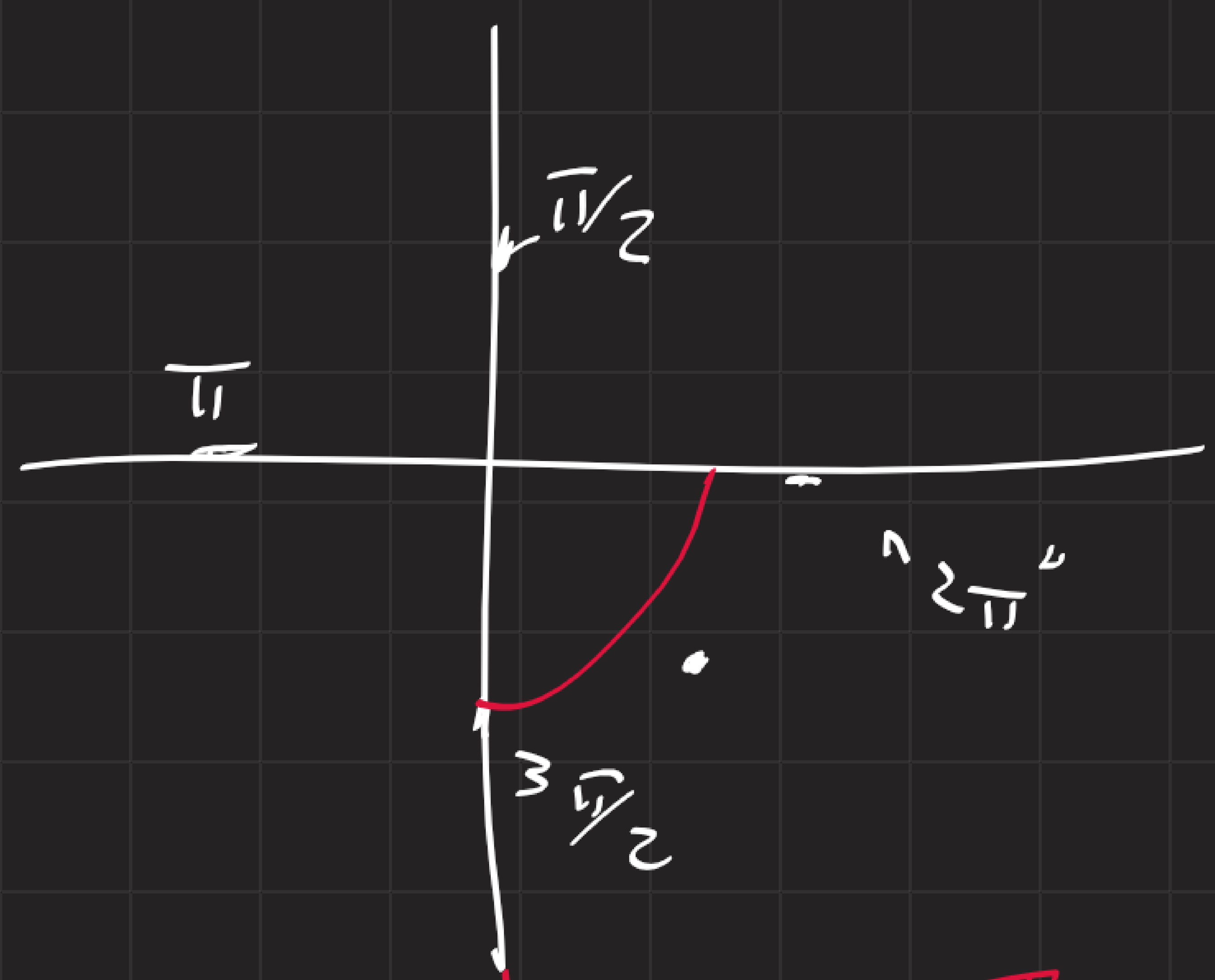
$= 16e^{i0}$ //

b) $\dfrac{(1-i)^{17}}{1+i^{17}}$

$i^{17} = i^{16}\cdot i$
$= (i^4)^4\cdot i$
$= 1^4\cdot i$
$= i$

$= \dfrac{(1-i)^{17}}{1+i} \cdot \dfrac{(1-i)}{(1-i)}$

$= \dfrac{(1-i)^{18}}{1+1} = \dfrac{1}{2}(1-i)^{18}$



$1-i \to |w| = \sqrt{2}$

$\to \tan(\theta) = \dfrac{-1}{1} \to \theta = \dfrac{-\pi}{4} + k\pi,\ k\in\mathbb{Z}$

con $k=2$, w está en IV.

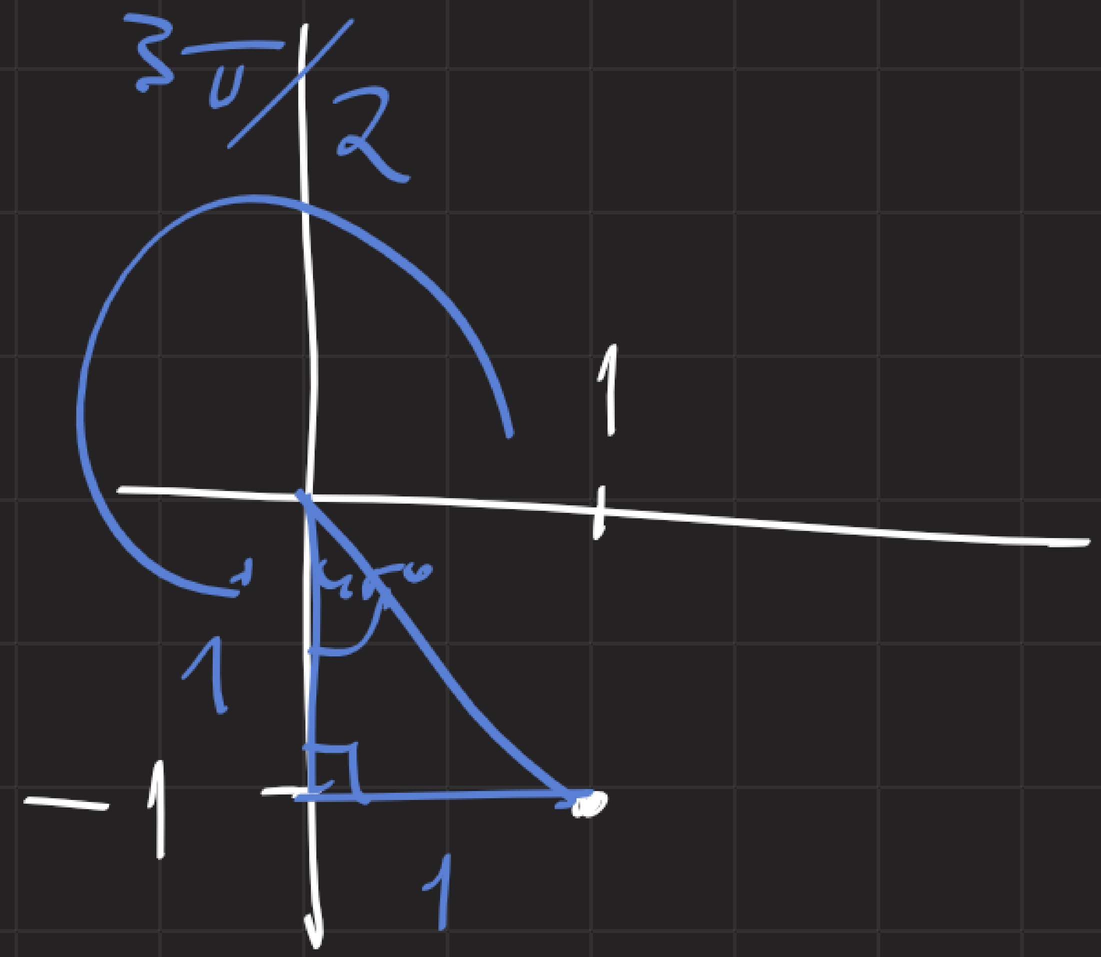
$w = \sqrt{2}\, e^{i\frac{7\pi}{4}}$

$\to = \dfrac{1}{2}\left(\left(\sqrt{2}\, e^{i\frac{7\pi}{4}}\right)^2\right)^9 = \dfrac{1}{2}\left(2e^{i\frac{7\pi}{2}}\right)^9$

$$\frac{1}{2}\left(2 e^{i\frac{7\pi}{2}}\right)^9 = \frac{1}{2}\cdot 2^9 \cdot e^{i\frac{63\pi}{2}} = 2^8 e^{i\frac{3\pi}{2}}$$

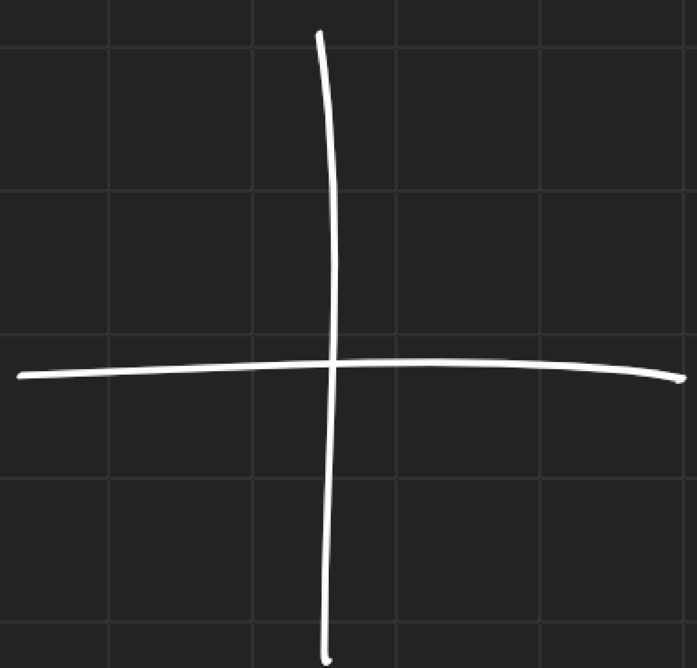
$$\frac{63\pi}{2} - 2k\pi$$

$$= 2\pi\Big(\underbrace{\underbrace{\tfrac{63}{4} - k}_{15{,}75}}_{\in[0,1)}\Big) \qquad k = 15$$

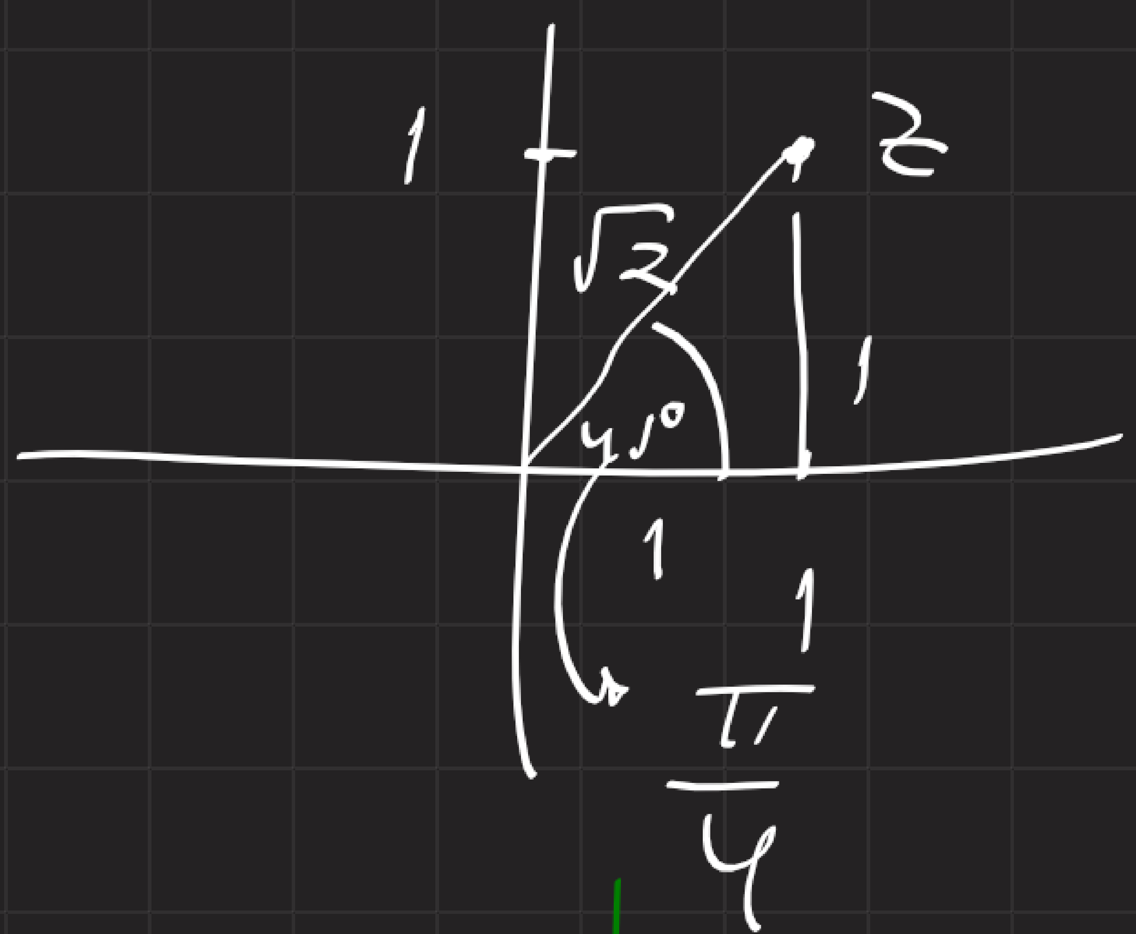


$$= 2\pi\left(\frac{63}{4} - \frac{60}{4}\right) = 2\pi\left(\frac{3}{4}\right) = \frac{3\pi}{2}$$

$$= 2^8\left(\cos\frac{3\pi}{2} + i \operatorname{sen}\frac{3\pi}{2}\right)$$



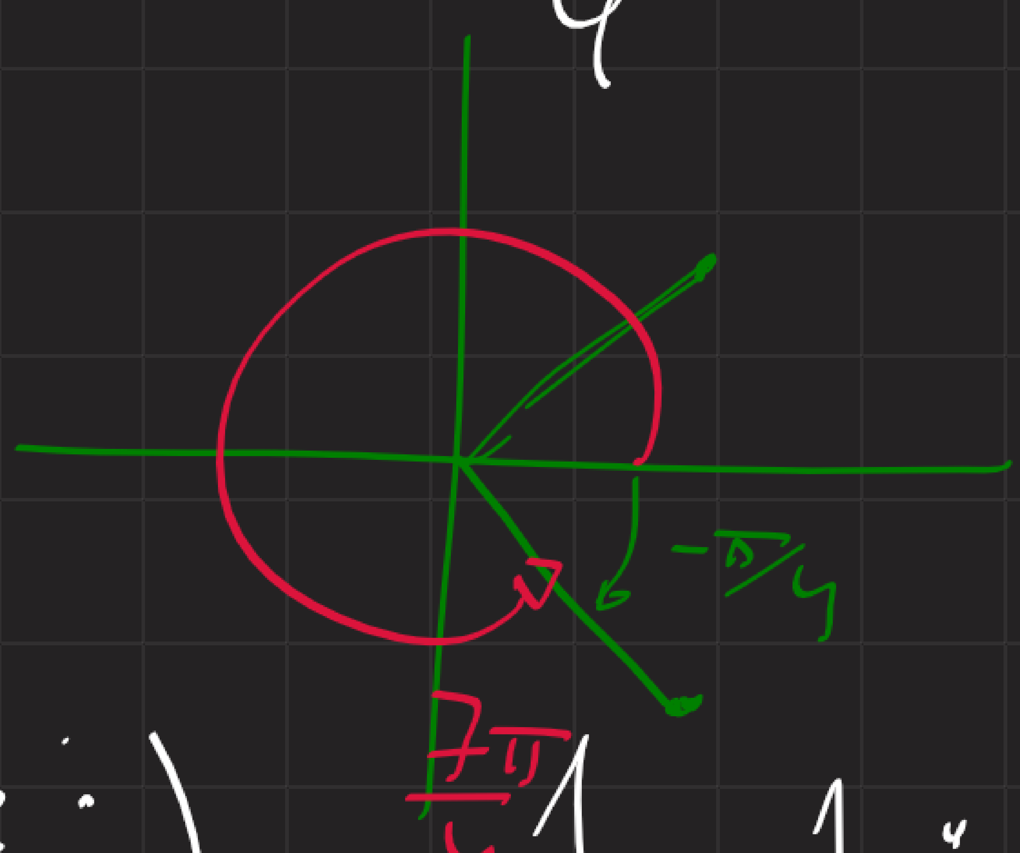
(c) $\frac{1}{1+i}$ $\quad z = 1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$



$$\Rightarrow \frac{1}{1+i} = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{-1}$$

$$= \frac{1}{\sqrt{2}}\, e^{i\frac{-\pi}{4}} \qquad [0, 2\pi)$$

$$= \frac{1}{\sqrt{2}}\, e^{i\frac{7\pi}{4}} \checkmark$$



$$= \frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{4} + i \operatorname{sen}\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2} + \frac{-\sqrt{2}}{2} i\right) = \frac{1}{2} - \frac{1}{2} i \checkmark$$

P3) $z \in \mathbb{C}$, $|z| > 1$, $\operatorname{Im} z > 0$ $\qquad$ $\circledast$ $e^{i} = \cos 1 + i \operatorname{sen} 1$

pdg: $\operatorname{Im}\left(z + \frac{1}{z}\right) > 0$ $\qquad$ $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

$$\operatorname{Im}\left(z + \frac{1}{z}\right) = \operatorname{Im}\left(z + \frac{\bar{z}}{|z|^2}\right) \quad / \operatorname{Im}(\cdot) \text{ morf.}$$

$$= \operatorname{Im}(z) + \operatorname{Im}\left(\frac{\bar{z}}{|z|^2}\right) \qquad |z|^2 \in \mathbb{R}$$

$$= \operatorname{Im}(z) + \frac{1}{|z|^2} \operatorname{Im}(\bar{z}) \quad / \operatorname{Im}(\bar{z}) = -\operatorname{Im}(z)$$

$$= \operatorname{Im}(z) - \frac{1}{|z|^2} \operatorname{Im}(z)$$

$$= \underbrace{\operatorname{Im} z}_{>0} \underbrace{\left(1 - \frac{1}{|z|^2}\right)}_{>0} > 0 \qquad \blacksquare$$

$z = a + ib$

$w = c + id$

$|z| > 1$

$\Rightarrow |z|^2 > 1$

$\Rightarrow \frac{1}{|z|^2} < 1$

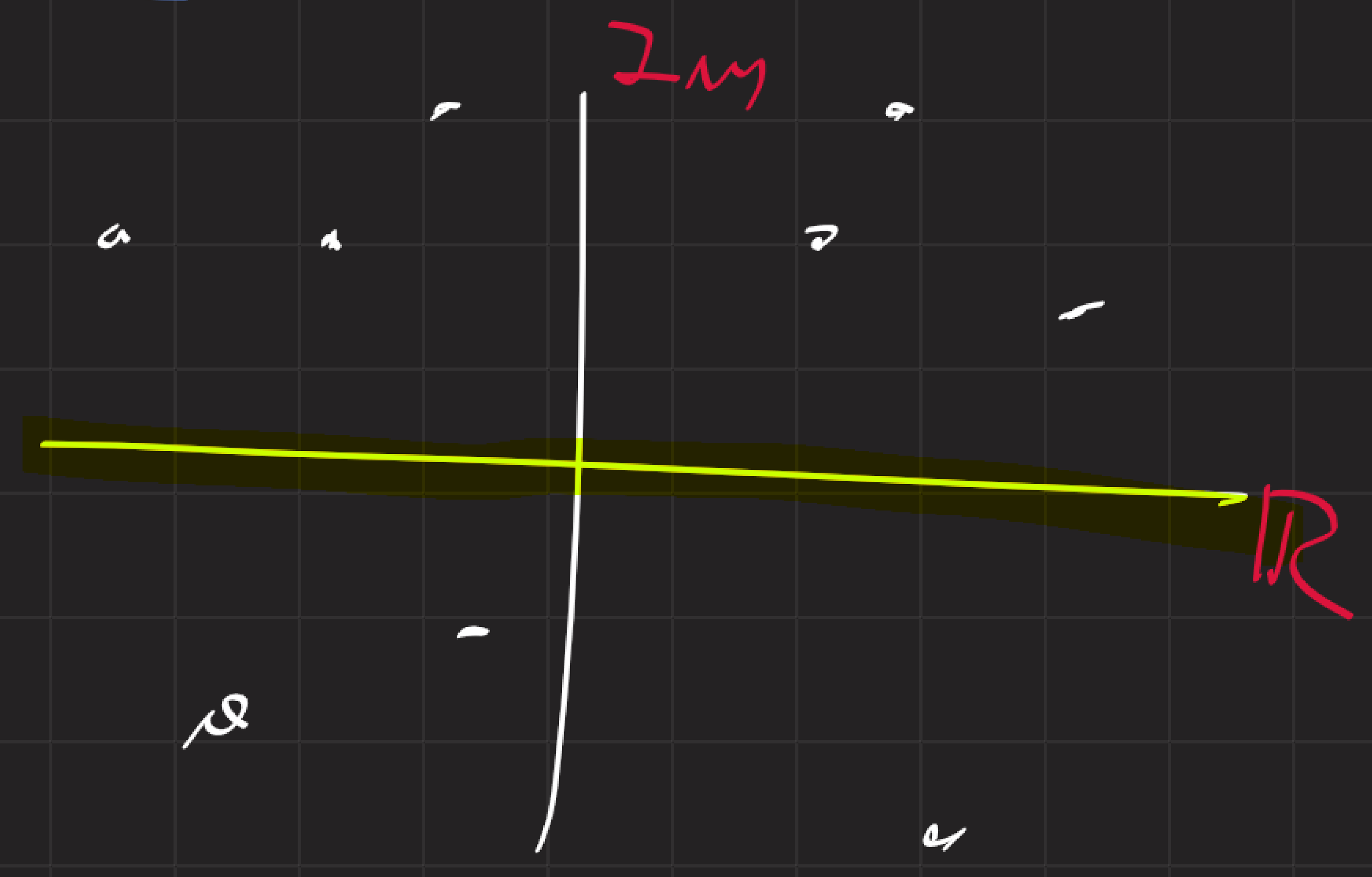
$\Rightarrow -\frac{1}{|z|^2} > -1 \quad / +1$

$\Rightarrow 1 - \frac{1}{|z|^2} > 0$

$$\operatorname{Im}(z + w)$$

$$= \operatorname{Im}(z) + \operatorname{Im}(w)$$

P3 | $z\in\mathbb{C}$ $\quad \boxed{|z|=1}$ $\quad z^{2n}\neq -1$



pdq: $\quad \dfrac{z^n}{1+z^{2n}}\in\mathbb{R}$

$$\Leftrightarrow \operatorname{Im}\left(\frac{z^n}{1+z^{2n}}\right)=0.$$

$$\operatorname{Im}\left(\frac{z^n}{1+z^{2n}}\right)=\frac{1}{2i}\left(\frac{z^n}{1+z^{2n}}-\overline{\left(\frac{z^n}{1+z^{2n}}\right)}\right)$$

$a\in\mathbb{R}\Rightarrow \bar a=a$. $\quad \overline{z\cdot w}=\bar z\cdot\bar w$ $\quad z\cdot\bar z=|z|^2$

$\overline{z+w}=\bar z+\bar w$ $\quad \overline{\left(\frac{z}{w}\right)}=\frac{\bar z}{\bar w}$

$\overline{(z^n)}=\bar z^n$

$$=\frac{1}{2i}\left(\frac{z^n}{1+z^{2n}}-\frac{\bar z^n}{1+\bar z^{2n}}\right)$$

$$z^n\cdot\bar z^{2n}=(z^n\cdot\bar z^n)\cdot\bar z^n=(|z|^2)^n\cdot\bar z^n$$

$$=\frac{1}{2i}\left(\frac{z^n(1+\bar z^{2n})-\bar z^n(1+z^{2n})}{\underbrace{(1+z^{2n})(1+\bar z^{2n})}_{w}}\right)$$

$$=\frac{1}{2iw}\left(z^n+z^n\cdot\bar z^{2n}-\bar z^n-\bar z^n\cdot z^{2n}\right)$$

$$=\frac{1}{2iw}\left(z^n+(|z|^2)^n\cdot\bar z^n-\bar z^n-(|z|^2)^n\cdot z^n\right)$$

$$=\frac{1}{2iw}\left(\cancel{z^n}+\cancel{\bar z^n}-\cancel{\bar z^n}-\cancel{z^n}\right)=0.$$

Py | a)

$$z^n = 1 \longrightarrow w_k = e^{i\frac{2\pi k}{n}},\ k \in \{0, \dots, n-1\}$$

$$\prod_{k=0}^{n-1} w_k = \prod_{n=0}^{n-1} e^{i\frac{2\pi k}{n}}$$

$$e^a \cdot e^b = e^{a+b}$$

$$\prod_i e^{a_i} = e^{\sum_i a_i}$$

$$= e^{\sum_{k=0}^{n-1} i\frac{2\pi k}{n}}$$

$$\sum_{i=0}^{n} i = \frac{n(n-1)}{2}$$

$$\sum_{n=0}^{n-1} i\frac{2\pi k}{n} = \frac{i2\pi}{n}\sum_{k=0}^{n-1} k$$

$$= i\frac{\cancel{2}\pi}{\cancel{n}}\frac{(n-1)\cancel{n}}{\cancel{2}} = i\pi(n-1)$$

$$= e^{i\pi(n-1)} = \left(e^{i\pi}\right)^{n-1} = (-1)^{n-1}$$

$$e^{ab} = (e^a)^b$$

b) $z \neq 1 \in \mathbb{C}$ una raíz cúbica de la unidad. Pruebe:

$$(1+z)^3 + (1+z^2)^9 + (1+z^3)^6 = 62.$$

z es raíz cúbica de 1

$$\Rightarrow \cancel{z_1 = 1} \lor z_2 = e^{i\frac{2\pi}{3}} \lor z_3 = e^{i\frac{4\pi}{3}}$$

$z \neq 1$.

$\underline{z = z_2}$ $z_1 + z_2 + z_3 = 0 \longrightarrow 1 + z_3 = -z_2$

$$= 1 + z_2 = -z_3$$

$$(1+z_2)^3 = (-z_3)^3 = -(z_3^3)$$
$$= -1$$

$$z_2^2 = \left(e^{i\frac{2\pi}{3}}\right)^2 = e^{i\frac{4\pi}{3}} = z_3$$

$$1 + z_2^2 = 1 + z_3 = -z_2$$

Luego $(1+z_2^2)^9 = (-z_2)^9 = ((-z_2)^3)^3$
$$= -1$$

$$z_2^3 = 1$$

$$1 + z_2^3 = 1 + 1 = 2$$

$$(1 + z_2^3)^6 = 2^6 = 64.$$

Luego:

$$(1+z_2)^3 + (1+z_2^2)^9 + (1+z_2^3)^6$$

$$= -1 + -1 + 64 = 62.$$

$z = z_3$ (*) $1 + z_2 + z_3 = 0$

- $(1+z_2)^3 = (-z_3)^3 = -z_3^3 = -1$

- $z_3^2 = \left(e^{i\frac{4\pi}{3}}\right)^2 = e^{i\frac{8\pi}{3}}$

$\frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$

$$= e^{i\frac{2\pi}{3}}$$

$$= z_2$$

$$\Rightarrow (1+z_3^2)^9 = (1+z_2)^9 = (-z_3)^9$$

$$= -1$$

- $\left(1+z_3^3\right)^6$

$= (1+1)^6 = 2^6 = 64$

$\Rightarrow \left(1+z_3\right)^3 + \left(1+z_3^2\right)^9 + \left(1+z_3^3\right)^6$

$= -1 + -1 + 64$

$= 62.$

P5

P5. *[¿¿??¿¿a?¿??¿]:* Sean z_1, z_2 las soluciones de $z^2 - 2z + 2 = 0$. Demuestre que $\forall n \in \mathbb{N}$, $\forall \theta \in \mathbb{R}$ (salvo múltiplos enteros de n) se tiene:

$$\frac{(\cot(\theta) + z_1 - 1)^n - (\cot(\theta) + z_2 - 1)^n}{z_1 - z_2} = \sin(n\theta)(\csc(\theta))^n$$

$$z^2 - 2z + 2 = 0$$

$$\Leftrightarrow z^2 - 2z + 1 = -1$$

$$\Leftrightarrow (z-1)^2 = -1$$

$$\Leftrightarrow z - 1 = \pm i$$

$$\Rightarrow z_{1,2} = 1 \pm i$$

SPG: $z_1 = 1 + i$, $z_2 = 1 - i$

$$\cot\theta = \frac{\cos\theta}{\operatorname{sen}\theta}$$

Reemplazando en LI.

$$\frac{(\cot\theta + 1 + i - 1)^n - (\cot(\theta) + 1 - i - 1)^n}{1 + i - 1 + i}$$

$$= \frac{\left(\frac{\cos\theta}{\operatorname{sen}\theta} + i\right)^n - \left(\frac{\cos\theta}{\operatorname{sen}\theta} - i\right)^n}{2i}$$

$$= \frac{\left(\frac{\cos\theta + i\operatorname{sen}\theta}{\operatorname{sen}\theta}\right)^n - \left(\frac{\cos\theta - i\operatorname{sen}\theta}{\operatorname{sen}\theta}\right)^n}{2i}$$

$$= \left[\left(\frac{e^{i\theta}}{\operatorname{sen}\theta}\right)^n - \left(\frac{e^{i(-\theta)}}{\operatorname{sen}\theta}\right)^n\right]\frac{1}{2i}$$

$$\frac{1}{2i\,\operatorname{sen}^{n}(\theta)}\left[e^{i\theta n}-e^{-i\theta n}\right]$$

$$=\frac{1}{2i\,\operatorname{sen}^{n}\theta}\left[\cancel{\cos n\theta}+i\operatorname{sen} n\theta-\cancel{\cos n\theta}+i\operatorname{sen} n\theta\right]$$

$$=\frac{1}{\cancel{2i}}\cdot\cancel{2i}\,\frac{\operatorname{sen} n\theta}{\operatorname{sen}^{n}\theta}=\csc^{n}\theta\cdot\operatorname{sen} n\theta$$