PAUTA AUX #8

P1)

(a) $$\sum_{j=1}^{n}\sum_{k=1}^{j}\left(k+\frac{2^j}{j}\right)$$

$$=\sum_{j=1}^{n}\left(\sum_{k=1}^{j}k+\sum_{k=1}^{j}\frac{2^j}{j}\right)$$

$$=\sum_{j=1}^{n}\left(\frac{j(j+1)}{2}+\frac{2^j}{\cancel{j}}\cdot\cancel{j}\right)$$

$$=\frac{1}{2}\sum_{j=1}^{n}j^2+\frac{1}{2}\sum_{j=1}^{n}j+\sum_{j=1}^{n}2^j$$

$$=\frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}\cdot\frac{n(n+1)}{2}+\frac{2^{n+1}-2^1}{2-1}$$

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(b) Desarrollemos el término de la sumatoria

$$\frac{1}{\sqrt{k(k+1)}\left(\sqrt{k+1}+\sqrt{k}\right)} \cdot \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}}$$

$$= \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}\left(\cancel{k}+1-\cancel{k}\right)}$$

$$= \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$$

Entonces, la sumatoria se escribe como

$$\sum_{k=1}^{n}\left[\underbrace{\frac{1}{\sqrt{k}}}_{a_k} - \underbrace{\frac{1}{\sqrt{k+1}}}_{a_{k+1}}\right] = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{n+1}}$$

donde se usó la propiedad telescópica //

(c)

$$\sum_{k=1}^{n} (k^2+1)\,k!$$

$$= \sum_{k=1}^{n} \left(k^2+2k+1-2k\right) k!$$

$$= \sum_{k=1}^{n} \left((k+1)^2-2k\right) k!$$

$$= \sum_{k=1}^{n} (k+1)\cdot(k+1)! - 2k\cdot k!$$

$$= \sum_{k=1}^{n} \Big[\underbrace{(k+1)(k+1)!}_{a_{k+1}} - \underbrace{k\,k!}_{a_k}\Big] - \sum_{k=1}^{n} k\,k!$$

$$= (n+1)(n+1)! - 1\cdot 1! - \sum_{k=1}^{n} k\,k! \qquad (i)$$

Notamos que

$$\sum_{k=1}^{n} k\,k! = \sum_{k=1}^{n} (k+1-1)\,k!$$

$$= \sum_{k=1}^{n} \underbrace{(k+1)!}_{a_{k+1}} - \underbrace{k!}_{a_k}$$

$$= (n+1)! - 1!$$

Entonces,

$$(i) = (n+1)(n+1)! - 1 - \left((n+1)! - 1\right)$$

$$= (n+1)(n+1)! \cancel{-1} - (n+1)! \cancel{+1}$$

$$= n\cdot(n+1)!$$

(d) Usamos la separación de pares e impares

$$\sum_{i=1}^{2n} (-1)^i i^2$$

$$= \sum_{i=1}^{n} \underbrace{(-1)^{2i}}_{1} (2i)^2 + \sum_{i=1}^{n} \underbrace{(-1)^{2i-1}}_{-1} (2i-1)^2$$

$$= \sum_{i=1}^{n} 4i^2 - \sum_{i=1}^{n} (4i^2 - 4i + 1)$$

$$= \cancel{\sum_{i=1}^{n} 4i^2} - \cancel{\sum_{i=1}^{n} 4i^2} + 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

$$= 4\,\frac{n(n+1)}{2} - 1\cdot(n - \cancel{1} + \cancel{1})$$

$$= 2n(n+1) - n$$

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P2

$$\binom{n}{k}\binom{n-k}{i-k}$$

$$= \frac{n!}{k!\,\cancel{(n-k)!}} \cdot \frac{\cancel{(n-k)!}}{(i-k)!\,(n-\cancel{k}-i+\cancel{k})!}$$

$$= \frac{n!}{k!\,(i-k)!\,(n-i)!} \cdot \frac{i!}{i!}$$

$$= \frac{n!}{i!\,(n-i)!} \cdot \frac{i!}{k!\,(i-k)!}$$

$$= \binom{n}{i}\binom{i}{k}$$

Con este resultado, tenemos que

$$\sum_{k=0}^{n}\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k}$$

$$= \sum_{k=0}^{n}\sum_{i=k}^{n}\binom{n}{k}\binom{n-k}{i-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\sum_{i=k}^{n}\binom{n-k}{i-k}$$

cambio de índices

$$= \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}\cdot 1^{i}\cdot 1^{n-k-i}$$

$$= \sum_{k=0}^{n}\binom{n}{k}(1+1)^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}2^{n-k}\cdot 1^{k}$$

$$= (2+1)^{n} = 3^{n} \quad \blacksquare$$

P3)

$$\sum_{k=0}^{n} \frac{(-1)^k}{(k+1)(k+2)} \binom{n}{k}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)(k+2)} \frac{n!}{k!\,(n-k)!}$$

$$= \sum_{k=0}^{n} (-1)^k \cdot \frac{n!}{(k+2)!\,(n-k)!} \cdot \frac{(n+1)(n+2)}{(n+1)(n+2)}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{(n+2)!}{(k+2)!\,(n-k)!} \cdot \frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)} \underbrace{\sum_{k=0}^{n} \binom{n+2}{k+2} (-1)^k}_{*}$$

$$\circledast = \sum_{k=2}^{n+2} \binom{n+2}{k} (-1)^{k-2}$$

$$= \sum_{k=2}^{n+2} \binom{n+2}{k} \frac{(-1)^k}{\cancel{(-1)^2}}$$

$$= \sum_{k=0}^{n+2} \left[\binom{n+2}{k} (-1)^k \right] - \binom{n+2}{1} (-1)^1 - \binom{n+2}{0} (-1)^0$$

$$= \sum_{k=0}^{n+2} \binom{n+2}{k} (-1)^k \cdot (1)^{n+2-k} + (n+2) - 1$$

$$= \cancel{(-1+1)^{n+2}} + n + 2 - 1$$

$$= n + 1$$

Entonces, el resultado final es

$$\frac{1}{\cancel{(n+1)}(n+2)} \cdot \cancel{(n+1)} = \frac{1}{n+2}$$

P4) (a)

$$|(A\cup C)\times(B\cup D)|$$

$$= |A\cup C|\cdot|B\cup D|$$

$$= \left(|A|+|C|-|A\cap C|\right)\left(|B|+|D|-|B\cap D|\right)$$

(b)

$\forall X, Y \in \mathcal{C} \quad |X|=|Y|$

Entonces, todos los conjuntos de $\mathcal{C}$ tienen el mismo cardinal. Digamos que ese cardinal es $n\in\mathbb{N}$.

Como $\mathcal{C}$ es partición, se tiene que

$$\bigodot_{C\in\mathcal{C}} C = A$$

$\bigodot$ es unión disjunta, es decir, todos los conjuntos son disjuntos entre sí.

$$\Rightarrow \left| \dot{\bigcup_{C \in \mathcal{C}}} C \right| = |A|$$

$$\Rightarrow \sum_{C \in \mathcal{C}} |C| = |A|$$

$$\Rightarrow \sum_{C \in \mathcal{C}} n = |A|$$

$$\Rightarrow n \sum_{C \in \mathcal{C}} 1 = |A|$$

$$\Rightarrow n\,|\mathcal{C}| = |A|$$

$\therefore$ $|\mathcal{C}|$ divide a $|A|$ $\blacksquare$