P1. Calcule:

(a) $\sum_{j=1}^{n}\sum_{k=1}^{j}\left(k+\frac{2^j}{j}\right)$

$$\sum_{j=1}^{n}\sum_{k=1}^{j}\left(k+\frac{2^j}{j}\right)$$

$$=\sum_{j=1}^{n}\left(\sum_{k=1}^{j}k+\boxed{\sum_{k=1}^{j}\frac{2^j}{j}}\right)$$

→ j es cte.

→ ¡no depende de k!

$$=\sum_{j=1}^{n}\left(\frac{j(j+1)}{2}+\frac{2^j}{j}\sum_{k=1}^{j}1\right)$$

$$=\sum_{j=1}^{n}\left(\frac{j(j+1)}{2}+\frac{2^j}{\cancel{j}}\cdot\cancel{j}\right)$$

$$=\underbrace{\sum_{j=1}^{n}\frac{j(j+1)}{2}}+\underbrace{\sum_{j=1}^{n}2^j+2^0-2^0}$$

$$=\sum_{j=0}^{n}2^j-1=2^{n+1}-1-1$$

$$=\frac{1}{2}\left(\sum_{j=1}^{n}j^2+j\right)$$

$$\sum_{k=m}^{n}a_k\to\sum_{k=m+p}^{n+p}a_{k-p}$$

$$=\frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right)$$

$$=\frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right)+2^{n+1}-2 \blacksquare$$

(b) $\sum_{k=1}^{n} \frac{1}{\sqrt{k(k+1)}(\sqrt{k+1}+\sqrt{k})}$

$\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}}$

$\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1})}$

$$= \sum_{k=1}^{n} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)$$

Racionalizando por $\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}}$

$$= \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{n+1}} \quad \boxed{W}$$

$$= \frac{\cancel{k}+1-\cancel{k}}{\sqrt{k}\sqrt{k+1}(\sqrt{k+1}+\sqrt{k})}$$

$$\sum_{k=1}^{n} \left(\frac{1}{\sqrt{k(k+1)}(\sqrt{k+1}+\sqrt{k})} \right) \cdot \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}}$$

$$= \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}\,(\cancel{k+1-k})} = \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}} = \frac{\cancel{\sqrt{k+1}}}{\sqrt{k\cancel{(k+1)}}} - \frac{\cancel{\sqrt{k}}}{\sqrt{\cancel{k}(k+1)}}$$

$$= \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$$

(c) $\displaystyle\sum_{k=1}^{n}(k^2+1)k!$

$$\begin{cases} (k+1)! \\ (k+1)^2 = k^2+2k+1 \end{cases}$$

$$(k^2+1)k!$$

$$= (k^2+2k+1-2k)\,k!$$

$$= ((k+1)^2-2k)\,k!$$

$$= (k+1)^2 k! - 2k\cdot k!$$

$k!\cdot(k+1) = (k+1)!$

$$= \underbrace{\left[(k+1)(k+1)! - k\cdot k!\right]}_{\text{¡telescópico!}} - \underbrace{k\cdot k!}_{\text{ma se :(}}$$

Entonces: $\displaystyle\sum_{k=1}^{m}(k^2+1)k!$

$$= \underbrace{\sum_{k=1}^{m}\left[(k+1)(k+1)! - k\cdot k!\right]}_{=(m+1)(m+1)!-1\cdot 1!} - \underbrace{\sum_{k=1}^{m} k\cdot k!}_{(*)}$$

$(*)$: $\displaystyle\sum_{k=1}^{m} k\cdot k!$ $\quad k = k+1-1$

$(k+1)! = (k+1)k!$

$$= \sum_{k=1}^{m}(k+1-1)k! = \underbrace{\sum_{k=1}^{m}(k+1)! - k!}_{\text{¡telescópica!}} = (m+1)! - 1!$$

Luego, nuestra suma original es igual a

$$= (n+1)(n+1)! - 1 - ((n+1)! - 1)$$

$$= (n+1)(n+1)! \cancel{-1} - (n+1)! \cancel{+1}$$

$$= (n+1)!\,(n+\cancel{1} - \cancel{1}) = n\,(n+1)! \quad \blacksquare$$

(d) $\sum_{i=1}^{2n}(-1)^i i^2$

$(-1)^i \to$

$i=0$	1
$i=1$	-1
$i=2$	1
$\vdots$	
i par	1
i impar	-1

PARES 1

IMPARES -1

$$-1 \cdot 1^2 + 1 \cdot 2^2 + (-1)\,3^2 + 1 \cdot 4^2 + (-1)\cdot 5^2 + \cdots$$

$$\cdots (-1)\cdot(2n-1)^2 + 1 \cdot (2n)^2$$

IMPARES PARES

$$\sum_{i=1}^{2n}(-1)^i\, i^2 = \underbrace{\sum_{j=1}^{n}(-1)\cdot(2j-1)^2}_{-(1^2+3^2+5^2+\cdots+(2n-1)^2)} + \underbrace{\sum_{j=1}^{n}(1)\cdot(2j)^2}_{2^2+4^2\ 6^2+\cdots+(2n)^2}$$

$$= -\sum_{j=1}^{n} \cancel{4j^2} - 4j + 1 + \sum_{j=1}^{n}\cancel{4j^2}$$

$$= \sum_{j=1}^{n} 4j - 1 = 4\,\frac{n(n+1)}{2} - n = \cdots = n(2n+1)$$

P2. Sean $i, k, n \in \mathbb{N}$ tales que $0 \leq k \leq i \leq n$. Demuestre que:

$$\binom{n}{k}\binom{n-k}{i-k} = \binom{n}{i}\binom{i}{k}$$

$$\left\{ \binom{n}{k} = \frac{n!}{k!(n-k)!} \right.$$

$$\binom{n}{k}\binom{n-k}{i-k} = \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{(i-k)!(n-k-(i-k))!}$$

$$n-\cancel{k}-i+\cancel{k}$$

$$= \frac{n!}{k!\cancel{(n-k)!}} \cdot \frac{\cancel{(n-k)!}}{(i-k)!(n-i)!}$$

$$= \frac{n!}{k!(i-k)!(n-i)!} \cdot \frac{i!}{i!}$$

$$\binom{n}{i}\binom{i}{k} = \frac{n!}{i!(n-i)!} \cdot \frac{i!}{k!(i-k)!}$$

$$= \frac{n!}{i!(n-i)!} \cdot \frac{i!}{k!(i-k)!} = \binom{n}{i}\binom{i}{k} \quad \boxed{\text{Q.E.D.}}$$

y úselo para demostrar que:

$$\sum_{k=0}^{n}\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k} = 3^n$$

$$\sum_{k=0}^{n}\left[\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k}\right]$$

$$= \sum_{k=0}^{n}\left[\sum_{i=k}^{n}\binom{n}{k}\binom{n-k}{i-k}\right]$$

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\sum_{i=k}^{n}\binom{n-k}{i-k}$$

cambio índice: $-k \Rightarrow a_{i+k}$

$$= \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{(i+\cancel{k})-\cancel{k}} = \sum_{k=0}^{n}\binom{n}{k}\underbrace{\sum_{i=0}^{n-k}\binom{n-k}{i}\cdot 1^{i}\cdot 1^{n-k-i}}_{=(1+1)^{n-k}}$$

$n-k = \tilde{n}$

$$\sum_{i=0}^{\tilde{n}}\binom{\tilde{n}}{i}\cdot 1^{i}\cdot 1^{\tilde{n}-i}$$

$$= (1+1)^{\tilde{n}} = 2^{n-k}.$$

$$= \sum_{k=0}^{n}\binom{n}{k}2^{n-k}\cdot 1^{k}$$

$$= (2+1)^n = 3^n.$$

$$\underbrace{\sum_{i=m}^{n} a_i}_{a_m + a_{m+1} + \cdots a_n} = \underbrace{\sum_{i=m+p}^{n+p} a_{i-p}}_{a_{m+\cancel{p}-\cancel{p}} + a_{m+\cancel{p}+1-\cancel{p}} + \cdots + a_{n+\cancel{p}-\cancel{p}}}$$

P3. Calcule
$$\sum_{k=0}^{n} \frac{(-1)^k}{(k+1)(k+2)} \binom{n}{k}$$

$(n+1)\cdot n! = (n+1)!$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)(k+2)} \cdot \frac{n!}{k!\,(n-k)!}$$

$$\binom{n+2}{k+2}$$

$$= \sum_{k=0}^{n} (-1)^k \cdot \frac{n!}{(k+2)!\,(n-k)!} \cdot \frac{(n+1)(n+2)}{(n+1)(n+2)}$$

$\to$ no depende de k.

$\hookrightarrow (n+2)-(k+2)$

$$= \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} (-1)^k \cdot \underbrace{\frac{(n+2)!}{(k+2)!\,(n-k)!}}_{\binom{n+2}{k+2}}$$

$$= \frac{1}{(n+1)(n+2)} \sum_{k=0+2}^{n+2} (-1)^k \binom{n+2}{k+2}$$

Cambio de índice

-2.

$$= \frac{1}{(n+1)(n+2)} \sum_{k=2}^{n+2} (-1)^{k-2} \binom{n+2}{k}$$

$$(-1)^{k-2} = \frac{(-1)^k}{(-1)^2} = (-1)^k$$

$$= \frac{1}{(n+1)(n+2)} \left(\sum_{k=2}^{n+2} \binom{n+2}{k} (-1)^k + \binom{n+2}{0}(-1)^0 - \binom{n+2}{0}(-1)^0 + \binom{n+2}{1}(-1)^1 - \binom{n+2}{1}(-1)^1 \right.$$

$$= \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n+2} \binom{n+2}{k} (-1)^k - 1 + (n+2)$$

$$= \frac{1}{(m+1)(m+2)} \left(\sum_{k=0}^{m+2} \binom{m+2}{k} (-1)^k \, 1^{m+2-k} - 1 + m + 2 \right)$$

$$= \frac{1}{(m+1)(m+2)} \cdot \left[\underbrace{(-1+1)}_{0}{}^{m+2} \cancel{-1} + m + \overset{+1}{\cancel{2}} \right]$$

$$= \frac{m+1}{(m+1)(m+2)} = \frac{1}{m+2} \quad \blacksquare$$

P4. (a) Sean A, B y C conjuntos finitos. Calcule la cardinalidad del siguiente conjunto, en términos del cardinal de A, B y C y/o sus intersecciones:

$$|(A \cup C) \times (B \cup C)|$$

$$|(A \cup C) \times (B \cup C)|$$

$$= |A \cup C| \cdot |B \cup C|$$

$$= \left(|A| + |C| - |A \cap C|\right) \cdot \left(|B| + |C| - |B \cap C|\right)$$

(b) Sea $\mathcal{C}$ una partición de un conjunto finito A de modo que para todo $X, Y \in \mathcal{C}$, $|X| = |Y|$. Demuestre que $|\mathcal{C}|$ divide a $|A|$.

(*)

$$|A| = m\,|\mathcal{C}|$$

(*) $\forall X, Y \in \mathcal{C}, \ |X| = |Y|.$

Todos tienen el mismo cardinal, digamos $\forall X \in \mathcal{C}, \ |X| = m$.
$m \in \mathbb{N}$.

¿pq son finitos los $X, \in \mathcal{C}$?

$\mathcal{C} \subseteq P(A)$

$\hookrightarrow$ A finito $\Rightarrow$ $P(A)$ finitos.

$(\mathcal{C} \subseteq \text{finito}) \Rightarrow \mathcal{C}$ es finito

$X \subseteq A$ finito $\Rightarrow$ X finito

$\forall X \in \mathcal{C} : |X| = m$

queremos demostrar que $|A| = m\,|\mathcal{C}|$

$$A = \bigcup_{B \in \mathcal{C}}^{\cdot} B \Rightarrow |A| = \left|\bigcup_{B \in \mathcal{C}}^{\cdot} B\right| = \sum_{B \in \mathcal{C}} |B| = \sum_{B \in \mathcal{C}} m$$

$$= m\,|\mathcal{C}|$$