# OPTIONS, FUTURES, AND OTHER DERIVATIVES

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CHAPTER 31

# Interest Rate Derivatives: Models of the Short Rate

The models for pricing interest rate options that we have presented so far make the assumption that the probability distribution of an interest rate, a bond price, or some other variable at a future point in time is lognormal. They are widely used for valuing instruments such as caps, European bond options, and European swap options. However, they have limitations. They do not provide a description of how interest rates evolve through time. Consequently, they cannot be used for valuing interest rate derivatives that are American-style or structured notes.

This chapter and the next discuss alternative approaches for overcoming these limitations. These involve building what is known as a *term structure model*. This is a model describing the evolution of all zero-coupon interest rates.[1] This chapter focuses on term structure models constructed by specifying the behavior of the short-term interest rate, $r$.

This chapter is concerned with modeling a single risk-free zero curve. The trend toward OIS discounting, discussed in Chapter 9, means that it is often necessary to model two zero curves simultaneously. The models in this chapter are then applied to the OIS rate and a separate model of the spread between OIS and LIBOR rates is developed. Section 32.3 discusses how this can be done.

## 31.1 BACKGROUND

The risk-free short rate, $r$, at time $t$ is the rate that applies to an infinitesimally short period of time at time $t$. It is sometimes referred to as the *instantaneous short rate*. Bond prices, option prices, and other derivative prices depend only on the process followed by $r$ in a risk-neutral world. The process for $r$ in the real world is not used. As explained in Chapter 28, the traditional risk-neutral world is a world where, in a very short time period between $t$ and $t + \Delta t$, investors earn on average $r(t)\Delta t$. All processes for $r$ that will be considered in this chapter, except where otherwise stated, are processes in this risk-neutral world.

[1] An advantage of term structure models is that the convexity and timing adjustments discussed in the previous chapter are not required.

From equation (28.19), the value at time $t$ of an interest rate derivative that provides a payoff of $f_T$ at time $T$ is

$$\hat{E}[e^{-\bar{r}(T-t)} f_T] \tag{31.1}$$

where $\bar{r}$ is the average value of $r$ in the time interval between $t$ and $T$, and $\hat{E}$ denotes expected value in the traditional risk-neutral world.

As usual, define $P(t, T)$ as the price at time $t$ of a risk-free zero-coupon bond that pays off \$1 at time $T$. From equation (31.1),

$$P(t, T) = \hat{E}[e^{-\bar{r}(T-t)}] \tag{31.2}$$

If $R(t, T)$ is the continuously compounded risk-free interest rate at time $t$ for a term of $T - t$, then

$$P(t, T) = e^{-R(t,T)(T-t)}$$

so that

$$R(t, T) = -\frac{1}{T-t}\ln P(t, T) \tag{31.3}$$

and, from equation (31.2),

$$R(t, T) = -\frac{1}{T-t}\ln \hat{E}[e^{-\bar{r}(T-t)}] \tag{31.4}$$

This equation enables the term structure of interest rates at any given time to be obtained from the value of $r$ at that time and the risk-neutral process for $r$. It shows that, once the process for $r$ has been defined, everything about the initial zero curve and its evolution through time can be determined.

Suppose $r$ follows the general process

$$dr = m(r, t)\,dt + s(r, t)\,dz$$

From Itô's lemma, any derivative dependent on $r$ follows the process

$$df = \left(\frac{\partial f}{\partial t} + m\frac{\partial f}{\partial r} + \tfrac{1}{2}s^2\frac{\partial f}{\partial r^2}\right)dt + s\frac{\partial f}{\partial r}\,dz$$

Because we are working in the traditional risk-neutral world, if the derivative provides no income, this process must have the form

$$df = rf\,dt + \cdots$$

so that

$$\frac{\partial f}{\partial t} + m\frac{\partial f}{\partial r} + \tfrac{1}{2}s^2\frac{\partial f}{\partial r^2} = rf \tag{31.5}$$

This is the equivalent of the Black–Scholes–Merton differential equation for interest rate derivatives. One particular solution to the equation must be the zero-coupon bond price $P(t, T)$.

## 31.2 EQUILIBRIUM MODELS

Equilibrium models usually start with assumptions about economic variables and derive a process for the short rate, $r$. They then explore what the process for $r$ implies about bond prices and option prices.

In a one-factor equilibrium model, the process for $r$ involves only one source of uncertainty. Usually the risk-neutral process for the short rate is described by an Itô process of the form

$$dr = m(r)\,dt + s(r)\,dz$$

The instantaneous drift, $m$, and instantaneous standard deviation, $s$, are assumed to be functions of $r$, but are independent of time. The assumption of a single factor is not as restrictive as it might appear. A one-factor model implies that all rates move in the same direction over any short time interval, but not that they all move by the same amount. The shape of the zero curve can therefore change with the passage of time.

This section considers three one-factor equilibrium models:

$m(r) = \mu r;\; s(r) = \sigma r$ (Rendleman and Bartter model)

$m(r) = a(b - r);\; s(r) = \sigma$ (Vasicek model)

$m(r) = a(b - r);\; s(r) = \sigma\sqrt{r}$ (Cox, Ingersoll, and Ross model)

## The Rendleman and Bartter Model

In Rendleman and Bartter's model, the risk-neutral process for $r$ is[2]

$$dr = \mu r\,dt + \sigma r\,dz$$

where $\mu$ and $\sigma$ are constants. This means that $r$ follows geometric Brownian motion. The process for $r$ is of the same type as that assumed for a stock price in Chapter 15. It can be represented using a binomial tree similar to the one used for stocks in Chapter 13.[3]

The assumption that the short-term interest rate behaves like a stock price is a natural starting point but is less than ideal. One important difference between interest rates and stock prices is that interest rates appear to be pulled back to some long-run average level over time. This phenomenon is known as *mean reversion*. When $r$ is high, mean reversion tends to cause it to have a negative drift; when $r$ is low, mean reversion tends to cause it to have a positive drift. Mean reversion is illustrated in Figure 31.1. The Rendleman and Bartter model does not incorporate mean reversion.

There are compelling economic arguments in favor of mean reversion. When rates are high, the economy tends to slow down and there is low demand for funds from borrowers. As a result, rates decline. When rates are low, there tends to be a high demand for funds on the part of borrowers and rates tend to rise.

## The Vasicek Model

In Vasicek's model, the risk-neutral process for $r$ is

$$dr = a(b - r)\,dt + \sigma\,dz$$

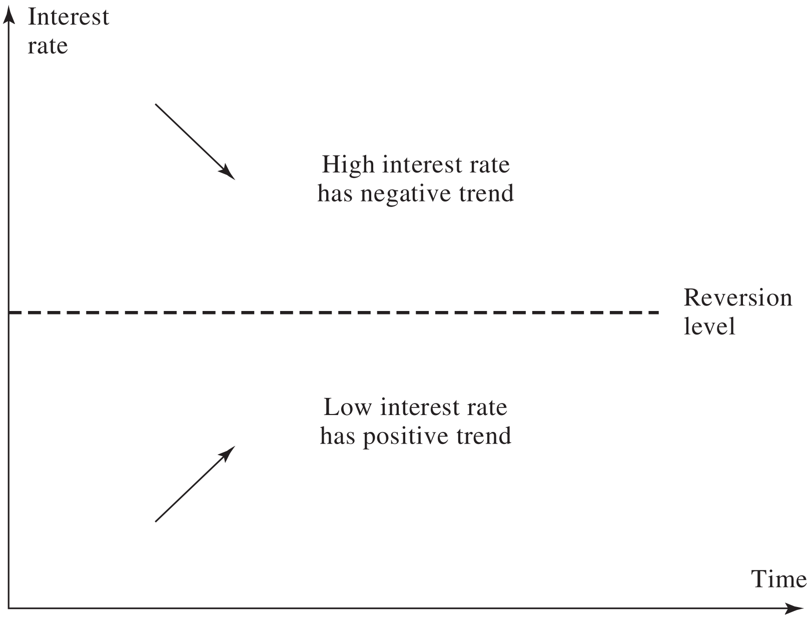
where $a$, $b$, and $\sigma$ are nonnegative constants.[4] This model incorporates mean reversion.

[2] See R. Rendleman and B. Bartter, "The Pricing of Options on Debt Securities," *Journal of Financial and Quantitative Analysis*, 15 (March 1980): 11–24.

[3] The way that the interest rate tree is used is explained later in the chapter.

[4] See O. A. Vasicek, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5 (1977): 177–88.

**Figure 31.1** Mean reversion.




The short rate is pulled to a level $b$ at rate $a$. Superimposed upon this "pull" is a normally distributed stochastic term $\sigma\, dz$.

Zero-coupon bond prices in Vasicek's model are given by

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)} \tag{31.6}$$

where

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a} \tag{31.7}$$

and

$$A(t, T) = \exp\left[\frac{(B(t, T) - T + t)(a^2 b - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t, T)^2}{4a}\right] \tag{31.8}$$

When $a = 0$, $B(t, T) = T - t$ and $A(t, T) = \exp[\sigma^2 (T - t)^3/6]$.

To see this, note that $m = a(b - r)$ and $s = \sigma$ in differential equation (31.5), so that

$$\frac{\partial f}{\partial t} + a(b - r)\frac{\partial f}{\partial r} + \tfrac{1}{2}\sigma^2 \frac{\partial f}{\partial r^2} = rf$$

By substitution, we see that $f = A(t, T) \exp^{-B(t,T)r}$ satisfies this differential equation when

$$B_t - aB + 1 = 0$$

and

$$A_t - abAB + \tfrac{1}{2}\sigma^2 AB^2 = 0$$

where subscripts denote derivatives. The expressions for $A(t, T)$ and $B(t, T)$ in equations (31.7) and (31.8) are solutions to these equations. What is more, because $A(T, T) = 1$ and $B(T, T) = 0$, the boundary condition $P(T, T) = 1$ is satisfied.

## The Cox, Ingersoll, and Ross Model

Cox, Ingersoll, and Ross (CIR) have proposed the following alternative model:[5]

$$dr = a(b - r)\,dt + \sigma\sqrt{r}\,dz$$

where $a$, $b$, and $\sigma$ are nonnegative constants. This has the same mean-reverting drift as Vasicek, but the standard deviation of the change in the short rate in a short period of time is proportional to $\sqrt{r}$. This means that, as the short-term interest rate increases, the standard deviation increases.

Bond prices in the CIR model have the same general form as those in Vasicek's model,

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)}$$

but the functions $B(t, T)$ and $A(t, T)$ are different:

$$B(t, T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}$$

and

$$A(t, T) = \left[\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{2ab/\sigma^2}$$

with $\gamma = \sqrt{a^2 + 2\sigma^2}$.

To see this result, we substitute $m = a(b - r)$ and $s = \sigma\sqrt{r}$ into differential equation (31.5) to get

$$\frac{\partial f}{\partial t} + a(b - r)\frac{\partial f}{\partial r} + \tfrac{1}{2}\sigma^2 r\frac{\partial f}{\partial r^2} = rf$$

As in the case of Vasicek's model, we can prove the bond-pricing result by substituting $f = A(t, T)e^{-B(t,T)r}$ into the differential equation. In this case, $A(t, T)$ and $B(t, T)$ are solutions of

$$B_t - aB - \tfrac{1}{2}\sigma^2 B^2 + 1 = 0, \quad A_t - abAB = 0$$

Furthermore, the boundary condition $P(T, T) = 1$ is satisfied.

## Properties of Vasicek and CIR

The $A(t, T)$ and $B(t, T)$ functions are different for Vasicek and CIR, but for both models

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)}$$

so that

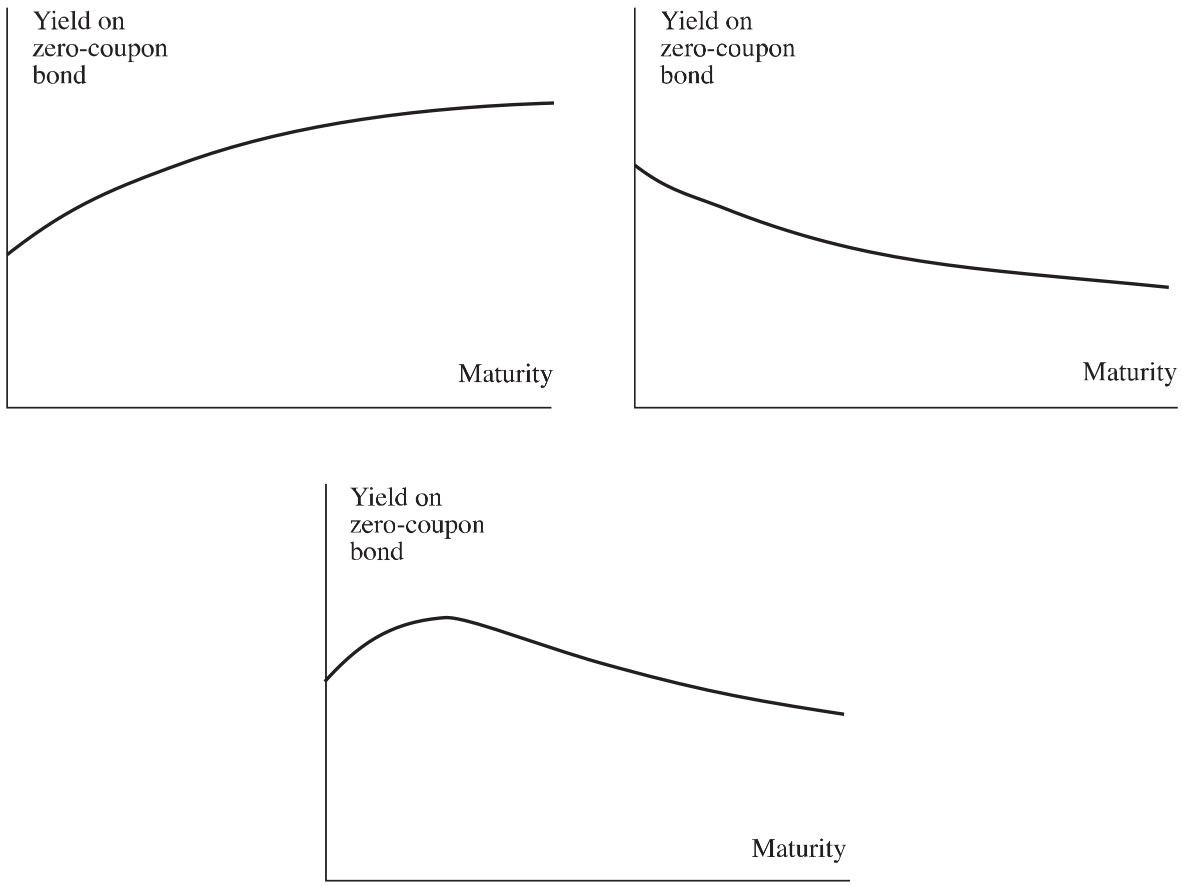
$$\frac{\partial P(t, T)}{\partial r(t)} = -B(t, T)P(t, T) \quad \textbf{(31.9)}$$

From equation (31.3), the zero rate at time $t$ for a period of $T - t$ is

$$R(t, T) = -\frac{1}{T - t}\ln A(t, T) + \frac{1}{T - t}B(t, T)r(t)$$

[5] See J.C. Cox, J.E. Ingersoll, and S.A. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53 (1985): 385–407.

**Figure 31.2** Possible shapes of term structure in the Vasicek and CIR models.




This shows that the entire term structure at time $t$ can be determined as a function of $r(t)$ once $a$, $b$, and $\sigma$ have been chosen. The rate $R(t, T)$ is linearly dependent on $r(t)$.[6] This means that the value of $r(t)$ determines the level of the term structure at time $t$. The shape of the term structure at time $t$ is independent of $r(t)$, but does depend on $t$. As shown in Figure 31.2, the shape at a particular time can be upward sloping, downward sloping, or slightly "humped."

In Chapter 4, we saw that the modified duration $D$ of a bond or other instrument dependent on interest rates, which has a price of $Q$, is defined by

$$\frac{\Delta Q}{Q} = -D\,\Delta y$$

where $y$ denotes the size of a parallel shift in the yield curve. An alternative duration measure $\hat{D}$, which can be used in conjunction with Vasicek or CIR, is defined as

[6] Some researchers have developed two-factor equilibrium models that give a richer set of possible movements in the term structure than either Vasicek or CIR. See, for example, F.A. Longstaff and E.S. Schwartz, "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model," *Journal of Finance*, 47, 4 (September 1992): 1259–82.

follows:

$$\hat{D} = -\frac{1}{Q}\frac{\partial Q}{\partial r}$$

When $Q$ is the zero-coupon bond, $P(t, T)$, equation (31.9) shows that $\hat{D} = B(t, T)$.

***Example 31.1***

Consider a zero-coupon bond lasting 4 years. In this case, $D = 4$, so that a 10-basis-point (0.1%) parallel shift in the term structure leads to a decrease of approximately 0.4% in the bond price. If Vasicek's model is used with $a = 0.1$,

$$\hat{D} = B(0, 4) = \frac{(1 - e^{-0.1\times 4})}{0.1} = 3.30$$

This means that a 10-basis-point increase in the short rate leads to a decrease in the bond price that is approximately 0.33%. The sensitivity of the bond price to movements in the short rate is less than to parallel shifts in the zero curve because of the impact of mean reversion.

When $Q$ is a portfolio of $n$ zero-coupon bonds, $P(t, T_i)$ $(1 \leqslant i \leqslant n)$, and $c_i$ is the principal of the $i$th bond, we have

$$\hat{D} = -\frac{1}{Q}\frac{\partial Q}{\partial r} = -\frac{1}{Q}\sum_{i=1}^{n} c_i \frac{\partial P(t, T_i)}{\partial r} = \sum_{i=1}^{n} \frac{c_i P(t, T_i)}{Q}\hat{D}_i$$

where $\hat{D}_i$ is the $\hat{D}$ for $P(t, T_i)$. This shows that the $\hat{D}$ for a coupon-bearing bond can be calculated as a weighted average of the $\hat{D}$'s for the underlying zero-coupon bonds, similarly to the way the usual duration measure $D$ is calculated (see Table 4.6). A convexity measure for Vasicek and CIR can be defined similarly to the duration measure (see Problem 31.21).

The expected growth rate of $P(t, T)$ in the traditional risk-neutral world at time $t$ is $r(t)$ because $P(t, T)$ is the price of a traded security. Since $P(t, T)$ is a function of $r(t)$, the coefficient of $dz(t)$ in the process for $P(t, T)$ can be calculated from Itô's lemma as $\sigma\,\partial P(t, T)/\partial r(t)$ for Vasicek and $\sigma\sqrt{r(t)}\,\partial P(t, T)/\partial r(t)$ for CIR. Substituting from equation (31.9), the processes for $P(t, T)$ in a risk-neutral world are therefore

$$\text{Vasicek:}\quad dP(t, T) = r(t)P(t, T)\,dt - \sigma B(t, T)P(t, T)\,dz(t)$$

$$\text{CIR:}\quad dP(t, T) = r(t)P(t, T)\,dt - \sigma\sqrt{r(t)}\,B(t, T)P(t, T)\,dz(t)$$

To compare the term structure of interest rates given by Vasicek and CIR for a particular value of $r$, it makes sense to use the same $a$ and $b$. However, the Vasicek $\sigma$, $\sigma_{\text{vas}}$, should be chosen to be approximately equal to the CIR $\sigma$, $\sigma_{\text{cir}}$, times $\sqrt{r(t)}$. For example, if $r$ is 4% and $\sigma_{\text{vas}} = 0.01$, an appropriate value for the $\sigma_{\text{cir}}$ would be $0.01/\sqrt{0.04} = 0.05$. Software for experimenting with the models can be found at www.rotman.utoronto.ca/~hull/VasicekCIR. Under Vasicek, $r$ can become negative. This is not possible under CIR.[7]

[7] In CIR, when interest rates get close to zero, the variability of interest rates becomes very small. In all circumstances, negative interest rates are not possible. Zero interest rates are not possible when $2ab \geqslant \sigma^2$.

## Applications of Equilibrium Models

As will be discussed in the next section, when derivatives are being valued it is important that the model used provides an exact fit to the current term structure of interest rates. However, when a Monte Carlo simulation is being carried out over a long period of time for the purposes of scenario analysis, the equilibrium models discussed in this section can be useful tools. A pension fund or insurance company that is interested in the value of its portfolio in 20 years is likely to feel that the precise shape of the current term structure of interest rates has relatively little bearing on its risks.

Once one of the models we have discussed has been chosen, one approach is to determine the parameters from past movements in the short-term interest rate. (The 1-month or 3-month rate can be used as a proxy for the short-term rate.) Data can be collected on daily, weekly, or monthly changes in the short rate and parameters can be estimated either by regressing $\Delta r$ against $r$ (see Example 31.2) or by using maximum-likelihood methods (see Problem 31.13). Another approach is to collect data on the prices of bonds and use an application such as Solver in Excel to determine the values of $a$, $b$, and $\sigma$ that minimize the sums of squares of the difference between the market prices of bonds and their model prices.

There is an important difference between the two approaches. The first approach (fitting historical data) provides parameter estimates in the real world. The second approach (fitting bond prices) provides parameter estimates in the risk-neutral world. When carrying out a scenario analysis, we are interested in modeling the behavior of the short rate in the real world. However, we are also likely to be interested in knowing the complete term structure of interest rates at different times during the life of the Monte Carlo simulation. For this we need risk-neutral parameter estimates.

When we move from the real world to the risk-neutral world, the volatility of the short rate does not change, but the drift does. To determine the change in the drift, it is necessary to make an estimate of the market price of interest rate risk. Ahmad and Wilmott do this by comparing the slope of the zero-coupon yield curve with the real-world drift of the short-term interest rate.[8] Their estimate of the long-term average market price of interest rate risk for US interest rates is about $-1.2$. There is a considerable variation in their estimate of the market price of interest rate risk through time. During stressed market conditions, when the "fear factor" is high (for example, during the 2007–2009 credit crisis), the market price of interest rate risk was found to be a much larger negative number than $-1.2$.

***Example 31.2***

Suppose that the discrete version of Vasicek's model

$$\Delta r = a(b - r)\Delta t + \sigma\epsilon\sqrt{\Delta t}$$

is used to fit weekly data on a short-term interest rate over a period of 10 years for the purposes of a Monte Carlo simulation. Assume that when $\Delta r$ (the change in the short rate in 1 week) is regressed against $r$, the slope is $-0.004$, the intercept is 0.00016, and the standard error of the estimate is 0.001. In this case, $\Delta t = 1/52$, so that $a/52 = 0.004$, $ab/52 = 0.00016$, and $\sigma/\sqrt{52} = 0.001$. This means that

[8] See R. Ahmad and P. Wilmott, "The Market Price of Interest-Rate Risk: Measuring and Modeling Fear and Greed in the Fixed-Income Markets," *Wilmott*, January 2007, 64–70.

$a = 0.21$, $b = 0.04$, and $\sigma = 0.0072$. (These parameters indicate that the short rate reverts to 4.0% with a reversion rate of 21%. The volatility of the short rate at any given time is 0.72% divided by the short rate.) The short rate can then be simulated in the real world.

To determine the risk-neutral process for $r$, we note that the proportional drift of $r$ is $a(b-r)/r$ and its volatility is $\sigma/r$. From the results in Chapter 28, the proportional drift reduces by $\lambda\sigma/r$ when we move from the real world to the risk-neutral world where $\lambda$ is the market price of interest rate risk. The process for $r$ in the risk-neutral world is therefore

$$dr = [a(b-r) - \lambda\sigma]\,dt + \sigma\,dz$$

or

$$dr = [a(b^* - r)]\,dt + \sigma\,dz$$

where

$$b^* = b - \lambda\sigma/a$$

Given the Ahmad and Wilmott results, we might choose to set $\lambda = -1.2$, so that $b^* = 0.04 + 1.2 \times 0.01/0.2 = 0.1$. Equations (31.6) to (31.8) (with $b = b^*$) can then be used to determine the complete term structure of interest rates at any point during the Monte Carlo simulation.

***Example 31.3***

The Cox–Ingersoll–Ross model

$$dr = a(b-r)\,dt + \sigma\sqrt{r}\,dz$$

can be used to value bonds of any maturity using the model's analytic results. Suppose that the values of $a$, $b$, and $\sigma$ that minimize the sum of the squared differences between the market prices of a set of bonds and the prices given by the model are $a = 0.15$, $b = 0.06$, and $\sigma = 0.05$. These values of the parameters give a best-fit risk-neutral process for the short-term interest rate. In this case, the proportional drift in the short rate is $a(b-r)/r$ and the volatility of the short rate $\sigma/\sqrt{r}$. From the results in Chapter 28, the proportional drift increases by $\lambda\sigma/\sqrt{r}$ when we move from the risk-neutral world to the real world where $\lambda$ is the market price of interest rate risk. The real-world process for $r$ is therefore

$$dr = [a(b-r) + \lambda\sigma\sqrt{r}]\,dt + \sigma\sqrt{r}\,dz$$

This can be used to simulate the process for the short rate in the real world.[9] At any given time longer rates can be determined using the risk-neutral process and analytic results. As before, we might choose to set $\lambda = -1.2$.

## 31.3 NO-ARBITRAGE MODELS

The disadvantage of the equilibrium models we have presented is that they do not automatically fit today's term structure of interest rates. By choosing the parameters judiciously, they can be made to provide an approximate fit to many of the term structures that are encountered in practice. But the fit is not an exact one. Most traders

[9] In moving between the real world and the risk-neutral world for the Cox–Ingersoll–Ross model, it can be convenient to assume that $\lambda$ is proportional to $\sqrt{r}$ or $1/\sqrt{r}$, so as to preserve the functional form for the drift.

find this unsatisfactory. Not unreasonably, they argue that they can have very little confidence in the price of a bond option when the model used does not price the underlying bond correctly. A 1% error in the price of the underlying bond may lead to a 25% error in an option price.

A *no-arbitrage model* is a model designed to be exactly consistent with today's term structure of interest rates. The essential difference between an equilibrium and a no-arbitrage model is therefore as follows. In an equilibrium model, today's term structure of interest rates is an output. In a no-arbitrage model, today's term structure of interest rates is an input.

In an equilibrium model, the drift of the short rate (i.e., the coefficient of $dt$) is not usually a function of time. In a no-arbitrage model, the drift is, in general, dependent on time. This is because the shape of the initial zero curve governs the average path taken by the short rate in the future in a no-arbitrage model. If the zero curve is steeply upward-sloping for maturities between $t_1$ and $t_2$, then $r$ has a positive drift between these times; if it is steeply downward-sloping for these maturities, then $r$ has a negative drift between these times.

It turns out that some equilibrium models can be converted to no-arbitrage models by including a function of time in the drift of the short rate. We now consider the Ho–Lee, Hull–White (one- and two-factor), Black–Derman–Toy, and Black–Karasinski models.

## The Ho–Lee Model

Ho and Lee proposed the first no-arbitrage model of the term structure in a paper in 1986.[10] They presented the model in the form of a binomial tree of bond prices with two parameters: the short-rate standard deviation and the market price of risk of the short rate. It has since been shown that the continuous-time limit of the model in the traditional risk-neutral world is

$$dr = \theta(t)\,dt + \sigma\,dz \tag{31.10}$$

where $\sigma$, the instantaneous standard deviation of the short rate, is constant and $\theta(t)$ is a function of time chosen to ensure that the model fits the initial term structure. The variable $\theta(t)$ defines the average direction that $r$ moves at time $t$. This is independent of the level of $r$. Ho and Lee's parameter that concerns the market price of risk is irrelevant when the model is used to price interest rate derivatives.

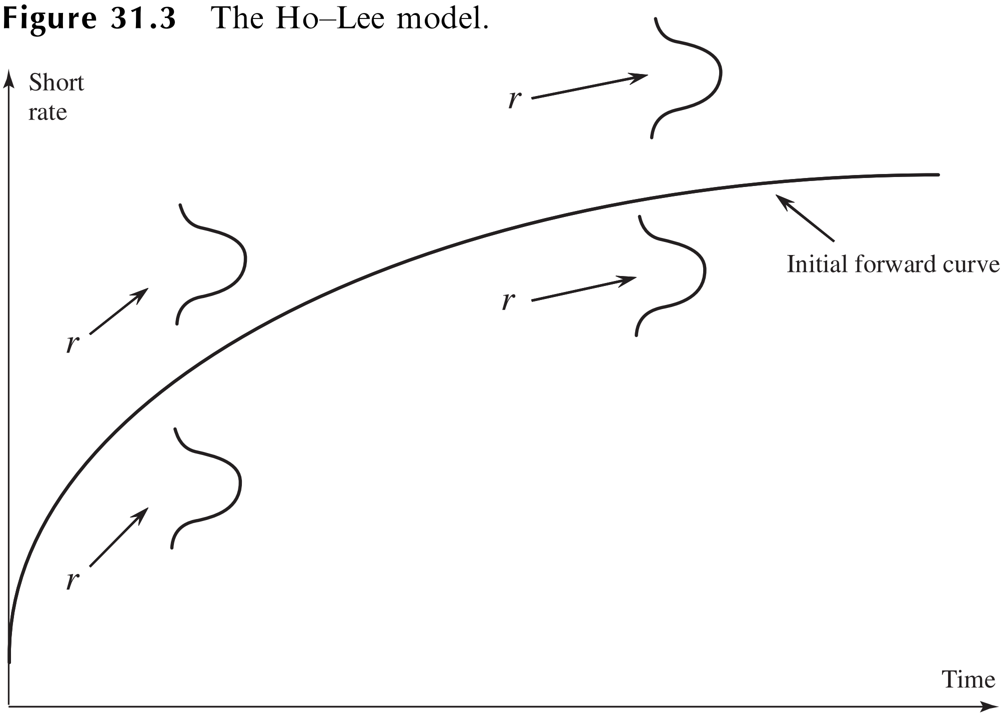
Technical Note 31 at www.rotman.utoronto.ca/~hull/TechnicalNotes shows that

$$\theta(t) = F_t(0, t) + \sigma^2 t \tag{31.11}$$

where $F(0, t)$ is the instantaneous forward rate for a maturity $t$ as seen at time zero and the subscript $t$ denotes a partial derivative with respect to $t$. As an approximation, $\theta(t)$ equals $F_t(0, t)$. This means that the average direction that the short rate will be moving in the future is approximately equal to the slope of the instantaneous forward curve. The Ho–Lee model is illustrated in Figure 31.3. Superimposed on the average movement in the short rate is the normally distributed random outcome.

[10] See T. S. Y. Ho and S.-B. Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 41 (December 1986): 1011–29.

**Figure 31.3** The Ho–Lee model.




Technical Note 31 also shows that

$$P(t, T) = A(t, T)e^{-r(t)(T-t)} \tag{31.12}$$

where

$$\ln A(t, T) = \ln \frac{P(0, T)}{P(0, t)} + (T - t)F(0, t) - \tfrac{1}{2}\sigma^2 t(T - t)^2$$

From Section 4.6, $F(0, t) = -\partial \ln P(0, t)/\partial t$. The zero-coupon bond prices, $P(0, t)$, are known for all $t$ from today's term structure of interest rates. Equation (31.12) therefore gives the price of a zero-coupon bond at a future time $t$ in terms of the short rate at time $t$ and the prices of bonds today.

## The Hull–White (One-Factor) Model

In a paper published in 1990, Hull and White explored extensions of the Vasicek model that provide an exact fit to the initial term structure.[11] One version of the extended Vasicek model that they consider is

$$dr = [\theta(t) - ar]\,dt + \sigma\,dz \tag{31.13}$$

or

$$dr = a\left[\frac{\theta(t)}{a} - r\right]dt + \sigma\,dz$$

where $a$ and $\sigma$ are constants. This is known as the Hull–White model. It can be characterized as the Ho–Lee model with mean reversion at rate $a$. Alternatively, it

[11] See J. Hull and A. White, "Pricing Interest Rate Derivative Securities," *Review of Financial Studies*, 3, 4 (1990): 573–92.

can be characterized as the Vasicek model with a time-dependent reversion level. At time $t$, the short rate reverts to $\theta(t)/a$ at rate $a$. The Ho–Lee model is a particular case of the Hull–White model with $a = 0$.

The model has the same amount of analytic tractability as Ho–Lee. Technical Note 31 shows that

$$\theta(t) = F_t(0, t) + aF(0, t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \quad \textbf{(31.14)}$$

The last term in this equation is usually fairly small. If we ignore it, the equation implies that the drift of the process for $r$ at time $t$ is $F_t(0, t) + a[F(0, t) - r]$. This shows that, on average, $r$ follows the slope of the initial instantaneous forward rate curve. When it deviates from that curve, it reverts back to it at rate $a$. The model is illustrated in Figure 31.4.

Technical Note 31 shows that bond prices at time $t$ in the Hull–White model are given by

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)} \quad \textbf{(31.15)}$$

where

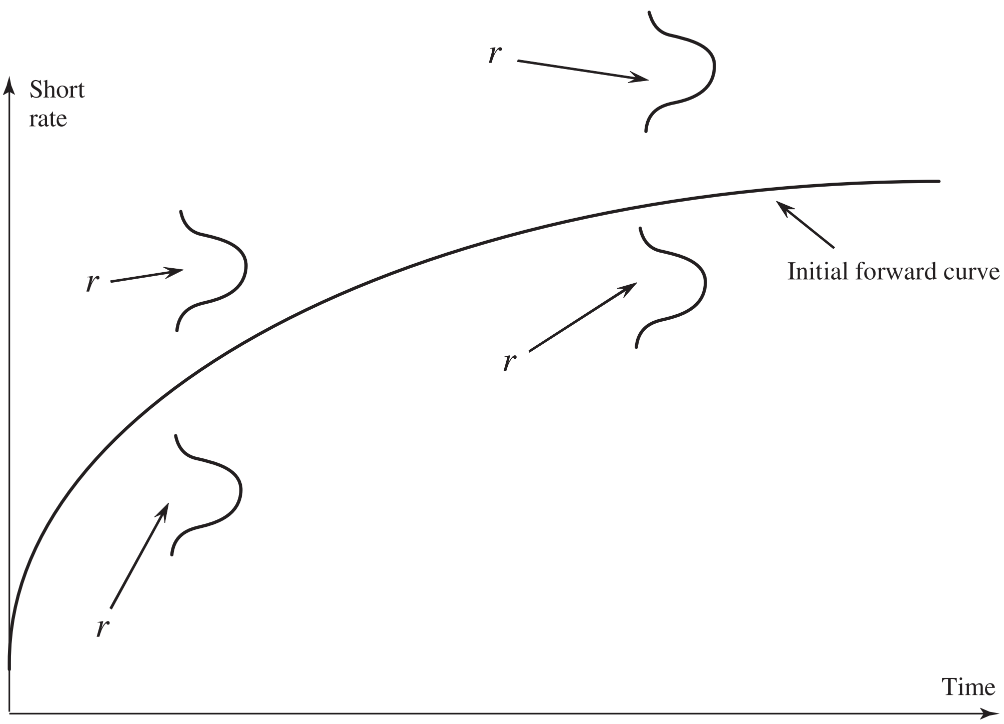
$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a} \quad \textbf{(31.16)}$$

and

$$\ln A(t, T) = \ln\frac{P(0, T)}{P(0, t)} + B(t, T)F(0, t) - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})^2(e^{2at} - 1) \quad \textbf{(31.17)}$$

As we show in the next section, European bond options can be valued analytically using the Ho–Lee and Hull–White models. A method for representing the models in the

**Figure 31.4** The Hull–White model.



form of a trinomial tree is given later in this chapter. This is useful when American options and other derivatives that cannot be valued analytically are considered.

## The Black–Derman–Toy Model

In 1990, Black, Derman, and Toy proposed a binomial-tree model for a lognormal short-rate process.[12] Their procedure for building the binomial tree is explained in Technical Note 23 at www.rotman.utoronto.ca/~hull/TechnicalNotes. It can be shown that the stochastic process corresponding to the model is

$$d\ln r = [\theta(t) - a(t)\ln r]\,dt + \sigma(t)\,dz$$

with

$$a(t) = -\frac{\sigma'(t)}{\sigma(t)}$$

where $\sigma'(t)$ is the derivative of $\sigma$ with respect to $t$. This model has the advantage over Ho–Lee and Hull–White that the interest rate cannot become negative. The Wiener process $dz$ can cause $\ln(r)$ to be negative, but $r$ itself is always positive. One disadvantage of the model is that there are no analytic properties. A more serious disadvantage is that the way the tree is constructed imposes a relationship between the volatility parameter $\sigma(t)$ and the reversion rate parameter $a(t)$. The reversion rate is positive only if the volatility of the short rate is a decreasing function of time.

In practice, the most useful version of the model is when $\sigma(t)$ is constant. The parameter $a$ is then zero, so that there is no mean reversion and the model reduces to

$$d\ln r = \theta(t)\,dt + \sigma\,dz$$

This can be characterized as a lognormal version of the Ho–Lee model.

## The Black–Karasinski Model

In 1991, Black and Karasinski developed an extension of the Black–Derman–Toy model where the reversion rate and volatility are determined independently of each other.[13] The most general version of the model is

$$d\ln r = [\theta(t) - a(t)\ln r]\,dt + \sigma(t)\,dz$$

The model is the same as Black–Derman–Toy model except that there is no relation between $a(t)$ and $\sigma(t)$. In practice, $a(t)$ and $\sigma(t)$ are often assumed to be constant, so that the model becomes

$$d\ln r = [\theta(t) - a\ln r]\,dt + \sigma\,dz \qquad \textbf{(31.18)}$$

As in the case of all the models we are considering, the $\theta(t)$ function is determined to provide an exact fit to the initial term structure of interest rates. The model has no analytic tractability, but later in this chapter we will describe a convenient way of

[12] See F. Black, E. Derman, and W. Toy, "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Prices," *Financial Analysts Journal*, January/February (1990): 33–39.

[13] See F. Black and P. Karasinski, "Bond and Option Pricing When Short Rates are Lognormal," *Financial Analysts Journal*, July/August (1991): 52–59.

simultaneously determining $\theta(t)$ and representing the process for $r$ in the form of a trinomial tree.

### The Hull–White Two-Factor Model

Hull and White have developed a two-factor model:[14]

$$df(r) = [\theta(t) + u - af(r)]\,dt + \sigma_1\,dz_1 \qquad \textbf{(31.19)}$$

where $f(r)$ is a function of $r$ and $u$ has an initial value of zero and follows the process

$$du = -bu\,dt + \sigma_2\,dz_2$$

As in the one-factor models just considered, the parameter $\theta(t)$ is chosen to make the model consistent with the initial term structure. The stochastic variable $u$ is a component of the reversion level of $f(r)$ and itself reverts to a level of zero at rate $b$. The parameters $a$, $b$, $\sigma_1$, and $\sigma_2$ are constants and $dz_1$ and $dz_2$ are Wiener processes with instantaneous correlation $\rho$.

This model provides a richer pattern of term structure movements and a richer pattern of volatilities than one-factor models of $r$. For more information on the analytical properties of the model and the way a tree can be constructed for it, see Technical Note 14 at www.rotman.utoronto.ca/~hull/TechnicalNotes.

## 31.4 OPTIONS ON BONDS

Some of the models just presented allow options on zero-coupon bonds to be valued analytically. For the Vasicek, Ho–Lee, and Hull–White one-factor models, the price at time zero of a call option that matures at time $T$ on a zero-coupon bond maturing at time $s$ is

$$LP(0, s)N(h) - KP(0, T)N(h - \sigma_P) \qquad \textbf{(31.20)}$$

where $L$ is the principal of the bond, $K$ is its strike price, and

$$h = \frac{1}{\sigma_P}\ln\frac{LP(0, s)}{P(0, T)K} + \frac{\sigma_P}{2}$$

The price of a put option on the bond is

$$KP(0, T)N(-h + \sigma_P) - LP(0, s)N(-h)$$

Technical Note 31 shows that, in the case of the Vasicek and Hull–White models,

$$\sigma_P = \frac{\sigma}{a}[1 - e^{-a(s-T)}]\sqrt{\frac{1 - e^{-2aT}}{2a}}$$

[14] See J. Hull and A. White, "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models," *Journal of Derivatives*, 2, 2 (Winter 1994): 37–48.

and, in the case of the Ho–Lee model,

$$\sigma_P = \sigma(s - T)\sqrt{T}$$

Equation (31.20) is essentially the same as Black's model for pricing bond options in Section 29.1 with the forward bond price volatility equaling $\sigma_P/\sqrt{T}$. As explained in Section 29.2, an interest rate cap or floor can be expressed as a portfolio of options on zero-coupon bonds. It can, therefore, be valued analytically using the equations just presented.

There are also formulas for valuing options on zero-coupon bonds in the Cox, Ingersoll, and Ross model, which we presented in Section 31.2. These involve integrals of the noncentral chi-square distribution.

### Options on Coupon-Bearing Bonds

In a one-factor model of $r$, all zero-coupon bonds move up in price when $r$ decreases and all zero-coupon bonds move down in price when $r$ increases. As a result, a one-factor model allows a European option on a coupon-bearing bond to be expressed as the sum of European options on zero-coupon bonds. The procedure is as follows:

1. Calculate $r^*$, the critical value of $r$ for which the price of the coupon-bearing bond equals the strike price of the option on the bond at the option maturity $T$.
2. Calculate prices of European options with maturity $T$ on the zero-coupon bonds that comprise the coupon-bearing bond. The strike prices of the options equal the values the zero-coupon bonds will have at time $T$ when $r = r^*$.
3. Set the price of the European option on the coupon-bearing bond equal to the sum of the prices on the options on zero-coupon bonds calculated in Step 2.

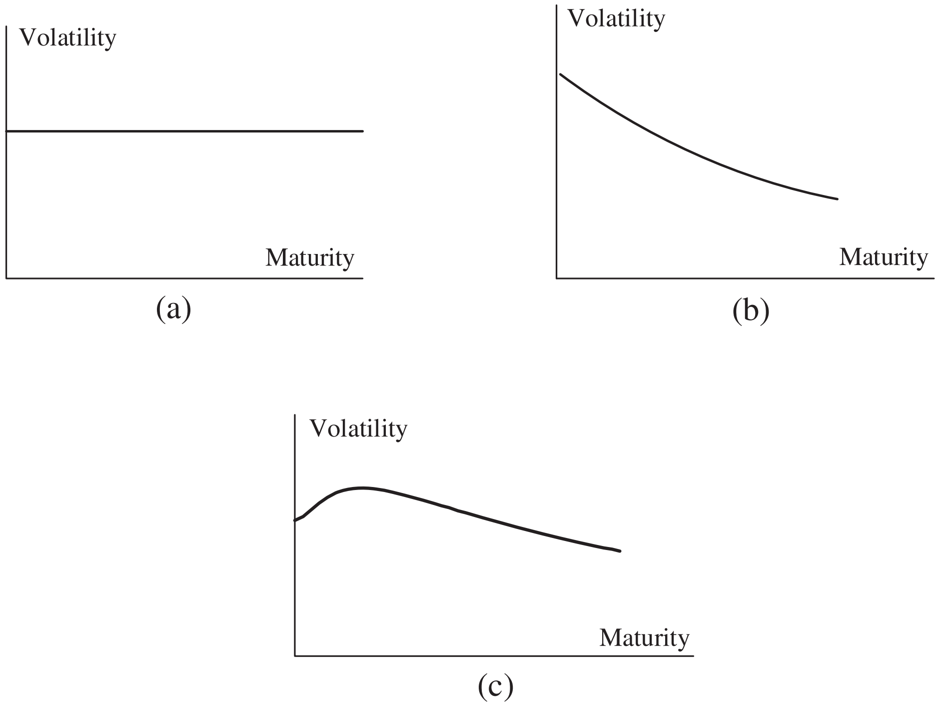
This allows options on coupon-bearing bonds to be valued for the Vasicek, Cox, Ingersoll, and Ross, Ho–Lee, and Hull–White models. As explained in Business Snapshot 29.2, a European swap option can be viewed as an option on a coupon-bearing bond. It can, therefore, be valued using this procedure. For more details on the procedure and a numerical example, see Technical Note 15 at www.rotman.utoronto.ca/~hull/TechnicalNotes.

## 31.5 VOLATILITY STRUCTURES

The models we have looked at give rise to different volatility environments. Figure 31.5 shows the volatility of the 3-month forward rate as a function of maturity for Ho–Lee, Hull–White one-factor and Hull–White two-factor models. The term structure of interest rates is assumed to be flat.

For Ho–Lee the volatility of the 3-month forward rate is the same for all maturities. In the one-factor Hull–White model the effect of mean reversion is to cause the volatility of the 3-month forward rate to be a declining function of maturity. In the Hull–White two-factor model when parameters are chosen appropriately, the volatility of the 3-month forward rate has a "humped" look. The latter is consistent with empirical evidence and implied cap volatilities discussed in Section 29.2.

**Figure 31.5** Volatility of 3-month forward rate as a function of maturity for (a) the Ho–Lee model, (b) the Hull–White one-factor model, and (c) the Hull–White two-factor model (when parameters are chosen appropriately).




## 31.6 INTEREST RATE TREES

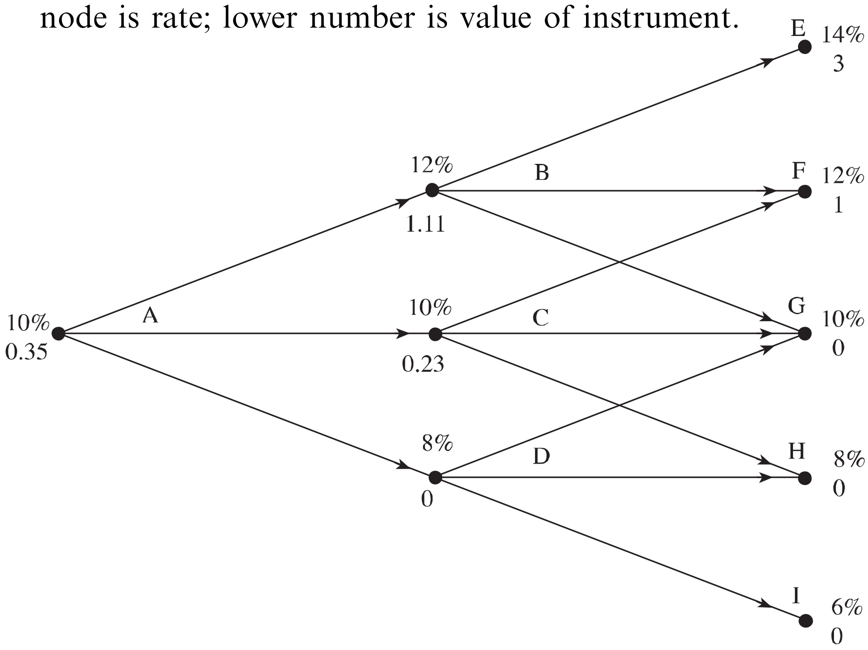
An interest rate tree is a discrete-time representation of the stochastic process for the short rate in much the same way as a stock price tree is a discrete-time representation of the process followed by a stock price. If the time step on the tree is $\Delta t$, the rates on the tree are the continuously compounded $\Delta t$-period rates. The usual assumption when a tree is constructed is that the $\Delta t$-period rate, $R$, follows the same stochastic process as the instantaneous rate, $r$, in the corresponding continuous-time model. The main difference between interest rate trees and stock price trees is in the way that discounting is done. In a stock price tree, the discount rate is usually assumed to be the same at each node or a function of time. In an interest rate tree, the discount rate varies from node to node.

It often proves to be convenient to use a trinomial rather than a binomial tree for interest rates. The main advantage of a trinomial tree is that it provides an extra degree of freedom, making it easier for the tree to represent features of the interest rate process such as mean reversion. As mentioned in Section 21.8, using a trinomial tree is equivalent to using the explicit finite difference method.

### Illustration of Use of Trinomial Trees

To illustrate how trinomial interest rate trees are used to value derivatives, consider the simple example shown in Figure 31.6. This is a two-step tree with each time step equal to 1 year in length so that $\Delta t = 1$ year. Assume that the up, middle, and down

**Figure 31.6** Example of the use of trinomial interest rate trees. Upper number at each node is rate; lower number is value of instrument.




probabilities are 0.25, 0.50, and 0.25, respectively, at each node. The assumed $\Delta t$-period rate is shown as the upper number at each node.[15]

The tree is used to value a derivative that provides a payoff at the end of the second time step of

$$\max[100(R - 0.11), 0]$$

where $R$ is the $\Delta t$-period rate. The calculated value of this derivative is the lower number at each node. At the final nodes, the value of the derivative equals the payoff. For example, at node E, the value is $100 \times (0.14 - 0.11) = 3$. At earlier nodes, the value of the derivative is calculated using the rollback procedure explained in Chapters 13 and 21. At node B, the 1-year interest rate is 12%. This is used for discounting to obtain the value of the derivative at node B from its values at nodes E, F, and G as

$$[0.25 \times 3 + 0.5 \times 1 + 0.25 \times 0]e^{-0.12\times 1} = 1.11$$

At node C, the 1-year interest rate is 10%. This is used for discounting to obtain the value of the derivative at node C as

$$(0.25 \times 1 + 0.5 \times 0 + 0.25 \times 0)e^{-0.1\times 1} = 0.23$$

At the initial node, A, the interest rate is also 10% and the value of the derivative is

$$(0.25 \times 1.11 + 0.5 \times 0.23 + 0.25 \times 0)e^{-0.1\times 1} = 0.35$$

## Nonstandard Branching

It sometimes proves convenient to modify the standard trinomial branching pattern that is used at all nodes in Figure 31.6. Three alternative branching possibilities are shown in

[15] We explain later how the probabilities and rates on an interest rate tree are determined.

**Figure 31.7** Alternative branching methods in a trinomial tree.

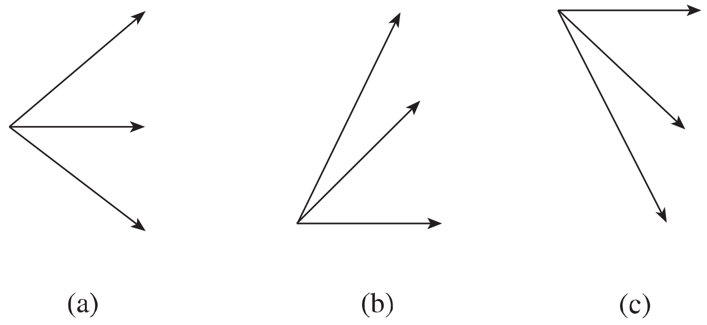



Figure 31.7. The usual branching is shown in Figure 31.7a. It is "up one/straight along/down one". One alternative to this is "up two/up one/straight along", as shown in Figure 31.7b. This proves useful for incorporating mean reversion when interest rates are very low. A third branching pattern shown in Figure 31.7c is "straight along/down one/down two". This is useful for incorporating mean reversion when interest rates are very high. The use of different branching patterns is illustrated in the following section.

## 31.7 A GENERAL TREE-BUILDING PROCEDURE

Hull and White have proposed a robust two-stage procedure for constructing trinomial trees to represent a wide range of one-factor models.[16] This section first explains how the procedure can be used for the Hull–White model in equation (31.13) and then shows how it can be extended to represent other models, such as Black–Karasinski.

### First Stage

The Hull–White model for the instantaneous short rate $r$ is

$$dr = [\theta(t) - ar]\,dt + \sigma\,dz$$

We suppose that the time step on the tree is constant and equal to $\Delta t$.[17]

Assume that the $\Delta t$ rate, $R$, follows the same process as $r$.

$$dR = [\theta(t) - aR]\,dt + \sigma\,dz$$

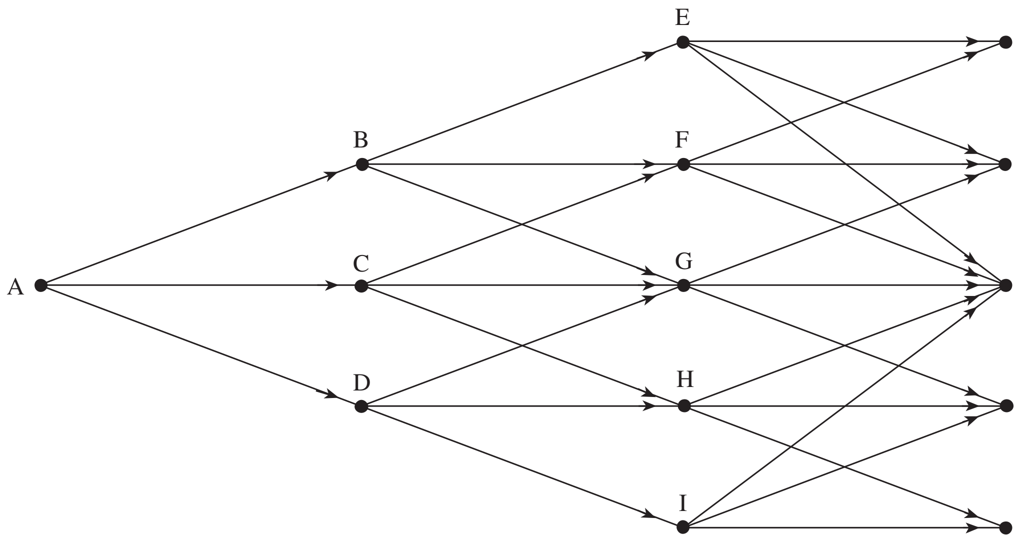
Clearly, this is reasonable in the limit as $\Delta t$ tends to zero. The first stage in building a tree for this model is to construct a tree for a variable $R^*$ that is initially zero and follows the process

$$dR^* = -aR^*\,dt + \sigma\,dz$$

[16] See J. Hull and A. White, "Numerical Procedures for Implementing Term Structure Models I: Single-Factor Models," *Journal of Derivatives*, 2, 1 (1994): 7–16; and J. Hull and A. White, "Using Hull–White Interest Rate Trees," *Journal of Derivatives*, (Spring 1996): 26–36.

[17] See Technical Note 16 at www.rotman.utoronto.ca/~hull/TechnicalNotes for a discussion of how nonconstant time steps can be used.

**Figure 31.8** Tree for $R^*$ in Hull–White model (first stage).




| *Node*: | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| $R^*$ (%) | 0.000 | 1.732 | 0.000 | −1.732 | 3.464 | 1.732 | 0.000 | −1.732 | −3.464 |
| $p_u$ | 0.1667 | 0.1217 | 0.1667 | 0.2217 | 0.8867 | 0.1217 | 0.1667 | 0.2217 | 0.0867 |
| $p_m$ | 0.6666 | 0.6566 | 0.6666 | 0.6566 | 0.0266 | 0.6566 | 0.6666 | 0.6566 | 0.0266 |
| $p_d$ | 0.1667 | 0.2217 | 0.1667 | 0.1217 | 0.0867 | 0.2217 | 0.1667 | 0.1217 | 0.8867 |

This process is symmetrical about $R^* = 0$. The variable $R^*(t + \Delta t) - R^*(t)$ is normally distributed. If terms of higher order than $\Delta t$ are ignored, the expected value of $R^*(t + \Delta t) - R^*(t)$ is $-aR^*(t)\Delta t$ and the variance of $R^*(t + \Delta t) - R^*(t)$ is $\sigma^2 \Delta t$.

The spacing between interest rates on the tree, $\Delta R$, is set as

$$\Delta R = \sigma\sqrt{3\Delta t}$$

This proves to be a good choice of $\Delta R$ from the viewpoint of error minimization.

The objective of the first stage of the procedure is to build a tree similar to that shown in Figure 31.8 for $R^*$. To do this, it is first necessary to resolve which of the three branching methods shown in Figure 31.7 will apply at each node. This will determine the overall geometry of the tree. Once this is done, the branching probabilities must also be calculated.

Define $(i, j)$ as the node where $t = i\,\Delta t$ and $R^* = j\,\Delta R$. (The variable $i$ is a positive integer and $j$ is a positive or negative integer.) The branching method used at a node must lead to the probabilities on all three branches being positive. Most of the time, the branching shown in Figure 31.7a is appropriate. When $a > 0$, it is necessary to switch from the branching in Figure 31.7a to the branching in Figure 31.7c for a sufficiently large $j$. Similarly, it is necessary to switch from the branching in Figure 31.7a to the branching in Figure 31.7b when $j$ is sufficiently negative. Define $j_{\max}$ as the value of $j$ where we switch from the Figure 31.7a branching to the Figure 31.7c branching and $j_{\min}$ as the value of $j$ where we switch from the Figure 31.7a branching to the Figure 31.7b branching. Hull and White show that probabilities are always positive if $j_{\max}$ is set equal to the smallest integer greater than $0.184/(a\,\Delta t)$ and $j_{\min}$ is set equal to $-j_{\max}$.[18] Define $p_u$, $p_m$, and $p_d$ as the probabilities of the highest, middle, and lowest branches emanating from the node. The probabilities are chosen to match the expected change and variance of the change in $R^*$ over the next time interval $\Delta t$. The probabilities must also sum to unity. This leads to three equations in the three probabilities.

As already mentioned, the mean change in $R^*$ in time $\Delta t$ is $-aR^*\Delta t$ and the variance of the change is $\sigma^2\Delta t$. At node $(i, j)$, $R^* = j\,\Delta R$. If the branching has the form shown in Figure 31.7a, the $p_u$, $p_m$, and $p_d$ at node $(i, j)$ must satisfy the following three equations to match the mean and standard deviation:

$$p_u\Delta R - p_d\Delta R = -aj\,\Delta R\,\Delta t$$
$$p_u\Delta R^2 + p_d\Delta R^2 = \sigma^2\Delta t + a^2j^2\Delta R^2\Delta t^2$$
$$p_u + p_m + p_d = 1$$

Using $\Delta R = \sigma\sqrt{3\Delta t}$, the solution to these equations is

$$p_u = \tfrac{1}{6} + \tfrac{1}{2}(a^2j^2\Delta t^2 - aj\,\Delta t)$$
$$p_m = \tfrac{2}{3} - a^2j^2\Delta t^2$$
$$p_d = \tfrac{1}{6} + \tfrac{1}{2}(a^2j^2\Delta t^2 + aj\,\Delta t)$$

Similarly, if the branching has the form shown in Figure 31.7b, the probabilities are

$$p_u = \tfrac{1}{6} + \tfrac{1}{2}(a^2j^2\Delta t^2 + aj\,\Delta t)$$
$$p_m = -\tfrac{1}{3} - a^2j^2\Delta t^2 - 2aj\,\Delta t$$
$$p_d = \tfrac{7}{6} + \tfrac{1}{2}(a^2j^2\Delta t^2 + 3aj\,\Delta t)$$

Finally, if the branching has the form shown in Figure 31.7c, the probabilities are

$$p_u = \tfrac{7}{6} + \tfrac{1}{2}(a^2j^2\Delta t^2 - 3aj\,\Delta t)$$
$$p_m = -\tfrac{1}{3} - a^2j^2\Delta t^2 + 2aj\,\Delta t$$
$$p_d = \tfrac{1}{6} + \tfrac{1}{2}(a^2j^2\Delta t^2 - aj\,\Delta t)$$

To illustrate the first stage of the tree construction, suppose that $\sigma = 0.01$, $a = 0.1$, and $\Delta t = 1$ year. In this case, $\Delta R = 0.01\sqrt{3} = 0.0173$, $j_{\max}$ is set equal to the smallest integer greater than 0.184/0.1, and $j_{\min} = -j_{\max}$. This means that $j_{\max} = 2$ and $j_{\min} = -2$ and the tree is as shown in Figure 31.8. The probabilities on the branches emanating from each node are shown below the tree and are calculated using the equations above for $p_u$, $p_m$, and $p_d$.

Note that the probabilities at each node in Figure 31.8 depend only on $j$. For example, the probabilities at node B are the same as the probabilities at node F. Furthermore, the tree is symmetrical. The probabilities at node D are the mirror image of the probabilities at node B.

---

[18] The probabilities are positive for any value of $j_{\max}$ between $0.184/(a\,\Delta t)$ and $0.816/(a\,\Delta t)$ and for any value of $j_{\min}$ between $-0.184/(a\,\Delta t)$ and $-0.816/(a\,\Delta t)$. Changing the branching at the first possible node proves to be computationally most efficient.

## Second Stage

The second stage in the tree construction is to convert the tree for $R^*$ into a tree for $R$. This is accomplished by displacing the nodes on the $R^*$-tree so that the initial term structure of interest rates is exactly matched. Define

$$\alpha(t) = R(t) - R^*(t)$$

The $\alpha(t)$'s that apply as the time step $\Delta t$ on the tree becomes infinitesimally small can be calculated analytically from equation (31.14).[19] However, we want a tree with a finite $\Delta t$ to match the term structure exactly. We therefore use an iterative procedure to determine the $\alpha$'s.

Define $\alpha_i$ as $\alpha(i\,\Delta t)$, the value of $R$ at time $i\,\Delta t$ on the $R$-tree minus the corresponding value of $R^*$ at time $i\,\Delta t$ on the $R^*$-tree. Define $Q_{i,j}$ as the present value of a security that pays off \$1 if node $(i, j)$ is reached and zero otherwise. The $\alpha_i$ and $Q_{i,j}$ can be calculated using forward induction in such a way that the initial term structure is matched exactly.

## Illustration of Second Stage

Suppose that the continuously compounded zero rates in the example in Figure 31.8 are as shown in Table 31.1. The value of $Q_{0,0}$ is 1.0. The value of $\alpha_0$ is chosen to give the right price for a zero-coupon bond maturing at time $\Delta t$. That is, $\alpha_0$ is set equal to the initial $\Delta t$-period interest rate. Because $\Delta t = 1$ in this example, $\alpha_0 = 0.03824$. This defines the position of the initial node on the $R$-tree in Figure 31.9. The next step is to calculate the values of $Q_{1,1}$, $Q_{1,0}$, and $Q_{1,-1}$. There is a probability of 0.1667 that the (1, 1) node is reached and the discount rate for the first time step is 3.82%. The value of $Q_{1,1}$ is therefore $0.1667e^{-0.0382} = 0.1604$. Similarly, $Q_{1,0} = 0.6417$ and $Q_{1,-1} = 0.1604$.

Once $Q_{1,1}$, $Q_{1,0}$, and $Q_{1,-1}$ have been calculated, $\alpha_1$ can be determined. It is chosen to give the right price for a zero-coupon bond maturing at time $2\Delta t$. Because $\Delta R = 0.01732$ and $\Delta t = 1$, the price of this bond as seen at node B is $e^{-(\alpha_1+0.01732)}$. Similarly, the price as

**Table 31.1** Zero rates for example in Figures 31.8 and 31.9.

| *Maturity* | *Rate (%)* |
|---|---|
| 0.5 | 3.430 |
| 1.0 | 3.824 |
| 1.5 | 4.183 |
| 2.0 | 4.512 |
| 2.5 | 4.812 |
| 3.0 | 5.086 |

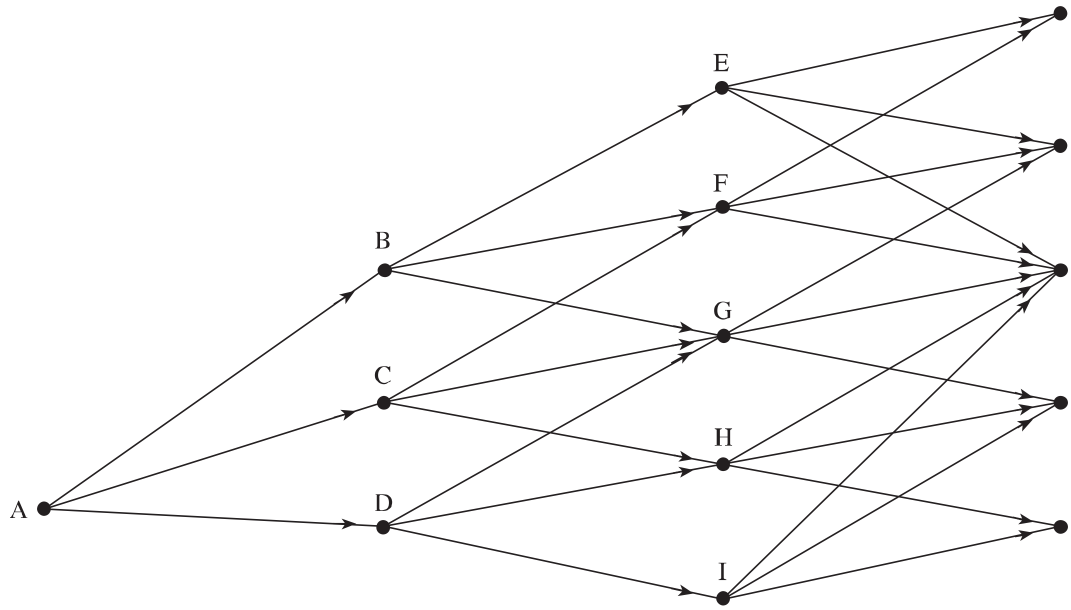
[19] To estimate the instantaneous $\alpha(t)$ analytically, we note that

$$dR = [\theta(t) - aR]\,dt + \sigma\,dz \quad \text{and} \quad dR^* = -aR^*\,dt + \sigma\,dz$$

so that $d\alpha = [\theta(t) - a\alpha(t)]\,dt$. Using equation (31.14), it can be seen that the solution to this is

$$\alpha(t) = F(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2.$$

**Figure 31.9** Tree for $R$ in Hull–White model (the second stage).




| *Node*: | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| $R$(%) | 3.824 | 6.937 | 5.205 | 3.473 | 9.716 | 7.984 | 6.252 | 4.520 | 2.788 |
| $p_u$ | 0.1667 | 0.1217 | 0.1667 | 0.2217 | 0.8867 | 0.1217 | 0.1667 | 0.2217 | 0.0867 |
| $p_m$ | 0.6666 | 0.6566 | 0.6666 | 0.6566 | 0.0266 | 0.6566 | 0.6666 | 0.6566 | 0.0266 |
| $p_d$ | 0.1667 | 0.2217 | 0.1667 | 0.1217 | 0.0867 | 0.2217 | 0.1667 | 0.1217 | 0.8867 |

seen at node C is $e^{-\alpha_1}$ and the price as seen at node D is $e^{-(\alpha_1-0.01732)}$. The price as seen at the initial node A is therefore

$$Q_{1,1}e^{-(\alpha_1+0.01732)} + Q_{1,0}e^{-\alpha_1} + Q_{1,-1}e^{-(\alpha_1-0.01732)} \tag{31.21}$$

From the initial term structure, this bond price should be $e^{-0.04512\times 2} = 0.9137$. Substituting for the $Q$'s in equation (31.21),

$$0.1604e^{-(\alpha_1+0.01732)} + 0.6417e^{-\alpha_1} + 0.1604e^{-(\alpha_1-0.01732)} = 0.9137$$

or

$$e^{-\alpha_1}(0.1604e^{-0.01732} + 0.6417 + 0.1604e^{0.01732}) = 0.9137$$

or

$$\alpha_1 = \ln\left[\frac{0.1604e^{-0.01732} + 0.6417 + 0.1604e^{0.01732}}{0.9137}\right] = 0.05205$$

This means that the central node at time $\Delta t$ in the tree for $R$ corresponds to an interest rate of 5.205% (see Figure 31.9).

The next step is to calculate $Q_{2,2}$, $Q_{2,1}$, $Q_{2,0}$, $Q_{2,-1}$, and $Q_{2,-2}$. The calculations can be shortened by using previously determined $Q$ values. Consider $Q_{2,1}$ as an example. This is the value of a security that pays off \$1 if node F is reached and zero otherwise. Node F can be reached only from nodes B and C. The interest rates at these nodes are 6.937% and 5.205%, respectively. The probabilities associated with the B–F and C–F branches are 0.6566 and 0.1667. The value at node B of a security that pays \$1 at node F is therefore $0.6566e^{-0.06937}$. The value at node C is $0.1667e^{-0.05205}$. The variable $Q_{2,1}$ is $0.6566e^{-0.06937}$ times the present value of \$1 received at node B plus $0.1667e^{-0.05205}$ times the present value of \$1 received at node C; that is,

$$Q_{2,1} = 0.6566e^{-0.06937} \times 0.1604 + 0.1667e^{-0.05205} \times 0.6417 = 0.1998$$

Similarly, $Q_{2,2} = 0.0182$, $Q_{2,0} = 0.4736$, $Q_{2,-1} = 0.2033$, and $Q_{2,-2} = 0.0189$.

The next step in producing the $R$-tree in Figure 31.9 is to calculate $\alpha_2$. After that, the $Q_{3,j}$'s can then be computed. The variable $\alpha_3$ can then be calculated, and so on.

## Formulas for $\alpha$'s and $Q$'s

To express the approach more formally, suppose that the $Q_{i,j}$ have been determined for $i \leqslant m$ ($m \geqslant 0$). The next step is to determine $\alpha_m$ so that the tree correctly prices a zero-coupon bond maturing at $(m+1)\,\Delta t$. The interest rate at node $(m, j)$ is $\alpha_m + j\,\Delta R$, so that the price of a zero-coupon bond maturing at time $(m+1)\Delta t$ is given by

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp[-(\alpha_m + j\,\Delta R)\Delta t] \tag{31.22}$$

where $n_m$ is the number of nodes on each side of the central node at time $m\,\Delta t$. The solution to this equation is

$$\alpha_m = \frac{\ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta R \Delta t} - \ln P_{m+1}}{\Delta t}$$

Once $\alpha_m$ has been determined, the $Q_{i,j}$ for $i = m+1$ can be calculated using

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k, j) \exp[-(\alpha_m + k\,\Delta R)\,\Delta t]$$

where $q(k, j)$ is the probability of moving from node $(m, k)$ to node $(m+1, j)$ and the summation is taken over all values of $k$ for which this is nonzero.

## Extension to Other Models

The procedure that has just been outlined can be extended to more general models of the form

$$df(r) = [\theta(t) - af(r)]\,dt + \sigma\,dz \tag{31.23}$$

where $f$ is a montonic function of $r$. This family of models has the property that they can fit any term structure.[20]

[20] Not all no-arbitrage models have this property. For example, the extended-CIR model, considered by Cox, Ingersoll, and Ross (1985) and Hull and White (1990), which has the form

$$dr = [\theta(t) - ar]\,dt + \sigma\sqrt{r}\,dz$$

cannot fit yield curves where the forward rate declines sharply. This is because the process is not well defined when $\theta(t)$ is negative.

As before, we assume that the $\Delta t$ period rate, $R$, follows the same process as $r$:

$$df(R) = [\theta(t) - af(R)]\,dt + \sigma\,dz$$

We start by setting $x = f(R)$, so that

$$dx = [\theta(t) - ax]\,dt + \sigma\,dz$$

The first stage is to build a tree for a variable $x^*$ that follows the same process as $x$ except that $\theta(t) = 0$ and the initial value is zero. The procedure here is identical to the procedure already outlined for building a tree such as that in Figure 31.8.

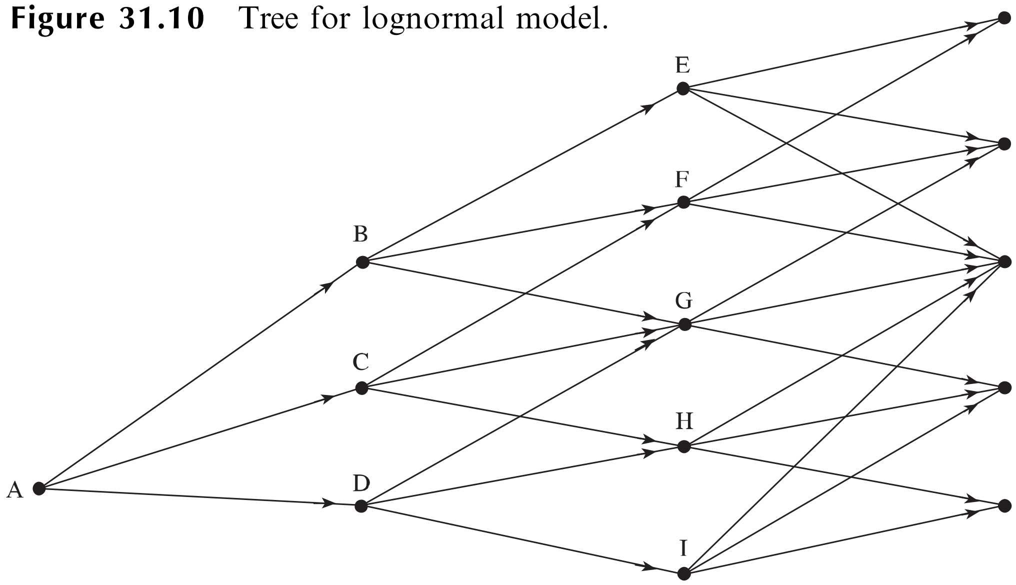
As in Figure 31.9, the nodes at time $i\,\Delta t$ are then displaced by an amount $\alpha_i$ to provide an exact fit to the initial term structure. The equations for determining $\alpha_i$ and $Q_{i,j}$ inductively are slightly different from those for the $f(R) = R$ case. The value of $Q$ at the first node, $Q_{0,0}$, is set equal to 1. Suppose that the $Q_{i,j}$ have been determined for $i \leqslant m$ ($m \geqslant 0$). The next step is to determine $\alpha_m$ so that the tree correctly prices an $(m+1)\Delta t$ zero-coupon bond. Define $g$ as the inverse function of $f$ so that the $\Delta t$-period interest rate at the $j$th node at time $m\,\Delta t$ is

$$g(\alpha_m + j\,\Delta x)$$

The price of a zero-coupon bond maturing at time $(m+1)\Delta t$ is given by

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp[-g(\alpha_m + j\,\Delta x)\Delta t] \qquad \textbf{(31.24)}$$

**Figure 31.10** Tree for lognormal model.




| *Node:* | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | −3.373 | −2.875 | −3.181 | −3.487 | −2.430 | −2.736 | −3.042 | −3.349 | −3.655 |
| $R$(%) | 3.430 | 5.642 | 4.154 | 3.058 | 8.803 | 6.481 | 4.772 | 3.513 | 2.587 |
| $p_u$ | 0.1667 | 0.1177 | 0.1667 | 0.2277 | 0.8609 | 0.1177 | 0.1667 | 0.2277 | 0.0809 |
| $p_m$ | 0.6666 | 0.6546 | 0.6666 | 0.6546 | 0.0582 | 0.6546 | 0.6666 | 0.6546 | 0.0582 |
| $p_d$ | 0.1667 | 0.2277 | 0.1667 | 0.1177 | 0.0809 | 0.2277 | 0.1667 | 0.1177 | 0.8609 |

This equation can be solved using a numerical procedure such as Newton–Raphson. The value $\alpha_0$ of $\alpha$ when $m = 0$, is $f\big(R(0)\big)$.

Once $\alpha_m$ has been determined, the $Q_{i,j}$ for $i = m + 1$ can be calculated using

$$Q_{m+1,j} = \sum_k Q_{m,k}q(k, j)\exp[-g(\alpha_m + k\,\Delta x)\Delta t]$$

where $q(k, j)$ is the probability of moving from node $(m, k)$ to node $(m + 1, j)$ and the summation is taken over all values of $k$ where this is nonzero.

Figure 31.10 shows the results of applying the procedure to the Black–Karasinski model in equation (31.18):

$$d\ln(r) = [\theta(t) - a\ln(r)]\,dt + \sigma\,dz$$

when $a = 0.22$, $\sigma = 0.25$, $\Delta t = 0.5$, and the zero rates are as in Table 31.1.

Setting $f(r) = r$ leads to the Hull–White model in equation (31.13); setting $f(r) = \ln(r)$ leads to the Black–Karasinksi model in equation (31.18). The main advantage of the $f(r) = r$ model is its analytic tractability. Its main disadvantage is that negative interest rates are possible. In many circumstances, the probability of negative interest rates occurring under the model is very small, but some analysts are reluctant to use a model where there is any chance at all of negative interest rates. The $f(r) = \ln r$ model has no analytic tractability, but has the advantage that interest rates are always positive.

## Handling Low Interest Rate Environments

When interest rates are very low, it is not easy to choose a satisfactory model. The probability of negative interest rates in the Hull–White model is no longer negligible. Also, the Black–Karasinski model does not work well because the same volatility is not appropriate for both low and high rates. One idea to avoid negative rates is to choose $f(r)$ as proportional to $\ln r$ when $r$ is low and proportional to $r$ when it is higher.[21] Another idea is to choose the short rate as the absolute value of the rate given by a Vasicek-type model. A better idea, suggested by Alexander Sokol, may be to construct a model where both the reversion rate and the volatility of $r$ are functions of $r$ estimated from empirical data. The variable $r$ can then be transformed to a new variable $x$ that has a constant $dz$ coefficient and the tree-building approach with more general trinomial branching than in Figure 31.7 can be used to implement the model.

## Using Analytic Results in Conjunction with Trees

When a tree is constructed for the $f(r) = r$ version of the Hull–White model, the analytic results in Section 31.3 can be used to provide the complete term structure and European option prices at each node. It is important to recognize that the interest rate on the tree is the $\Delta t$-period rate $R$. It is not the instantaneous short rate $r$.

From equations (31.15), (31.16), and (31.17) it can be shown (see Problem 31.20) that

$$P(t, T) = \hat{A}(t, T)e^{-\hat{B}(t,T)R} \tag{31.25}$$

[21] See J. Hull and A. White, "Taking Rates to the Limit," *Risk*, December (1997): 168–69.

where

$$\ln \hat{A}(t, T) = \ln \frac{P(0, T)}{P(0, t)} - \frac{B(t, T)}{B(t, t + \Delta t)} \ln \frac{P(0, t + \Delta t)}{P(0, t)} - \frac{\sigma^2}{4a}(1 - e^{-2at})B(t, T)[B(t, T) - B(t, t + \Delta t)] \quad \textbf{(31.26)}$$

and

$$\hat{B}(t, T) = \frac{B(t, T)}{B(t, t + \Delta t)} \Delta t \quad \textbf{(31.27)}$$

(In the case of the Ho–Lee model, we set $\hat{B}(t, T) = T - t$ in these equations.)

Bond prices should therefore be calculated with equation (31.25), and not with equation (31.15).

***Example 31.1***

Suppose zero rates are as in Table 31.2. The rates for maturities between those indicated are generated using linear interpolation.

Consider a 3-year ($= 3 \times 365$ days) European put option on a zero-coupon bond that will pay 100 in 9 years ($= 9 \times 365$ days). Interest rates are assumed to follow the Hull–White ($f(r) = r$) model. The strike price is 63, $a = 0.1$, and $\sigma = 0.01$. A 3-year tree is constructed and zero-coupon bond prices are calculated analytically at the final nodes as just described. As shown in Table 31.3, the results from the tree are consistent with the analytic price of the option.

This example provides a good test of the implementation of the model because the gradient of the zero curve changes sharply immediately after the expiration of the option. Small errors in the construction and use of the tree are liable to have a big effect on the option values obtained. (The example is used in Sample Application G of the DerivaGem Applications software.)

**Table 31.2** Zero curve with all rates continuously compounded, actual/365.

| *Maturity* | *Days* | *Rate* (%) |
|---|---|---|
| 3 days | 3 | 5.01772 |
| 1 month | 31 | 4.98284 |
| 2 months | 62 | 4.97234 |
| 3 months | 94 | 4.96157 |
| 6 months | 185 | 4.99058 |
| 1 year | 367 | 5.09389 |
| 2 years | 731 | 5.79733 |
| 3 years | 1,096 | 6.30595 |
| 4 years | 1,461 | 6.73464 |
| 5 years | 1,826 | 6.94816 |
| 6 years | 2,194 | 7.08807 |
| 7 years | 2,558 | 7.27527 |
| 8 years | 2,922 | 7.30852 |
| 9 years | 3,287 | 7.39790 |
| 10 years | 3,653 | 7.49015 |

**Table 31.3** Value of a three-year put option on a nine-year zero-coupon bond with a strike price of 63: $a = 0.1$ and $\sigma = 0.01$; zero curve as in Table 31.2.

| *Steps* | *Tree* | *Analytic* |
|---|---|---|
| 10 | 1.8468 | 1.8093 |
| 30 | 1.8172 | 1.8093 |
| 50 | 1.8057 | 1.8093 |
| 100 | 1.8128 | 1.8093 |
| 200 | 1.8090 | 1.8093 |
| 500 | 1.8091 | 1.8093 |

### Tree for American Bond Options

The DerivaGem software accompanying this book implements the normal and the lognormal model for valuing European and American bond options, caps/floors, and European swap options. Figure 31.11 shows the tree produced by the software when it is used to value a 1.5-year American call option on a 10-year bond using four time steps and the lognormal (Black–Karasinski) model. The parameters used in the lognormal model are $a = 5\%$ and $\sigma = 20\%$. The underlying bond lasts 10 years, has a principal of 100, and pays a coupon of 5% per annum semiannually. The yield curve is flat at 5% per annum. The strike price is 105. As explained in Section 29.1 the strike price can be a cash strike price or a quoted strike price. In this case it is a quoted strike price. The bond price shown on the tree is the cash bond price. The accrued interest at each node is shown below the tree. The cash strike price is calculated as the quoted strike price plus accrued interest. The quoted bond price is the cash bond price minus accrued interest. The payoff from the option is the cash bond price minus the cash strike price. Equivalently it is the quoted bond price minus the quoted strike price.

The tree gives the price of the option as 0.672. A much larger tree with 100 time steps gives the price of the option as 0.703. Note that the price of the 10-year bond cannot be computed analytically when the lognormal model is assumed. It is computed numerically by rolling back through a much larger tree than that shown.

## 31.8 CALIBRATION

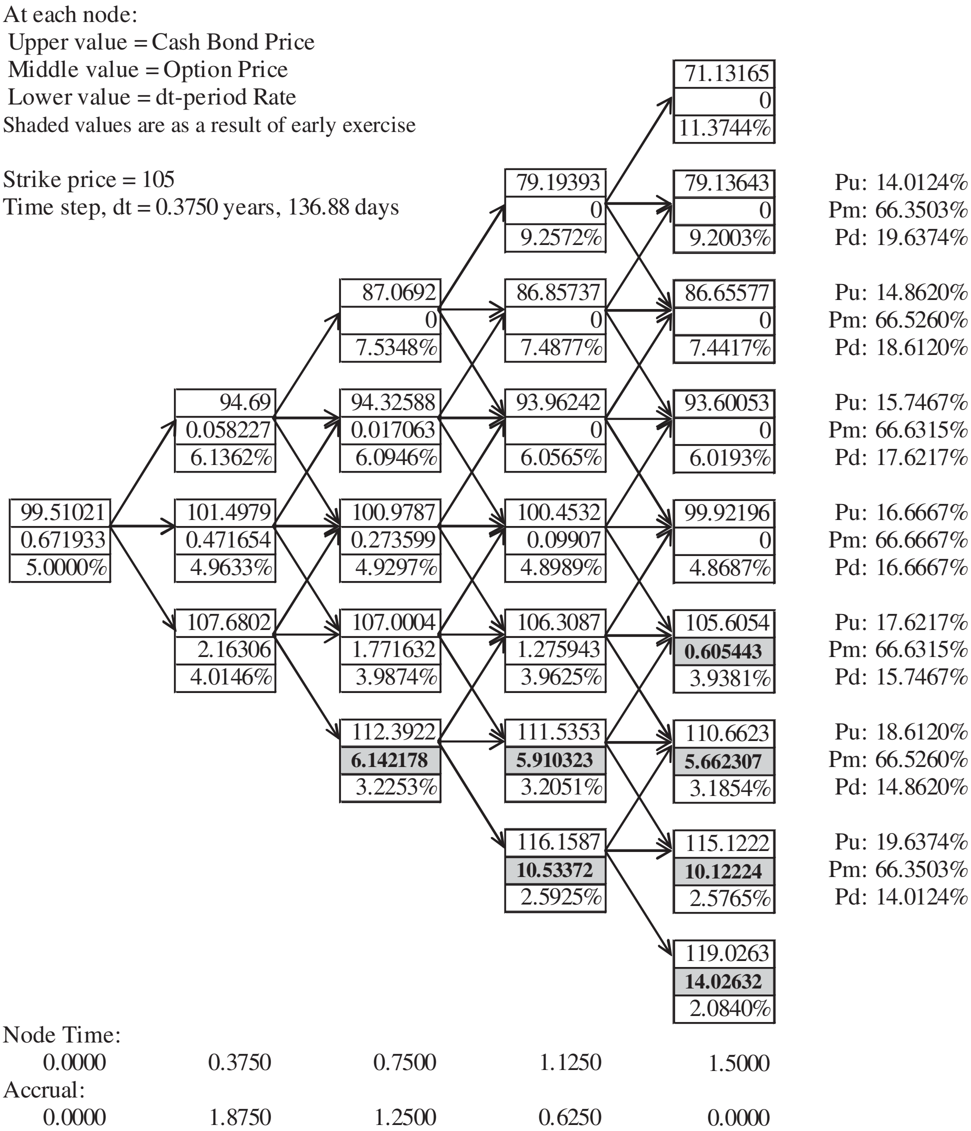
Up to now, we have assumed that the volatility parameters $a$ and $\sigma$ are known. We now discuss how they are determined. This is known as calibrating the model.

The volatility parameters are determined from market data on actively traded options (e.g., broker quotes on caps and swap options such as those in Tables 29.1 and 29.2). These will be referred to as the *calibrating instruments*. The first stage is to choose a "goodness-of-fit" measure. Suppose there are $n$ calibrating instruments. A popular goodness-of-fit measure is

$$\sum_{i=1}^{n}(U_i - V_i)^2$$

where $U_i$ is the market price of the $i$th calibrating instrument and $V_i$ is the price given by

**Figure 31.11** Tree, produced by DerivaGem, for valuing an American bond option.




the model for this instrument. The objective of calibration is to choose the model parameters so that this goodness-of-fit measure is minimized.

The number of volatility parameters should not be greater than the number of calibrating instruments. If $a$ and $\sigma$ are constant, there are only two volatility parameters. The models can be extended so that $a$ or $\sigma$, or both, are functions of time. Step functions can be used. Suppose, for example, that $a$ is constant and $\sigma$ is a function of time. We might choose times $t_1, t_2, \dots, t_n$ and assume $\sigma(t) = \sigma_0$ for $t \leqslant t_1$, $\sigma(t) = \sigma_i$ for $t_i < t \leqslant t_{i+1}$ $(1 \leqslant i \leqslant n-1)$, and $\sigma(t) = \sigma_n$ for $t > t_n$. There would then be a total of $n+2$ volatility parameters: $a$, $\sigma_0$, $\sigma_1, \dots,$ and $\sigma_n$.

The minimization of the goodness-of-fit measure can be accomplished using the Levenberg–Marquardt procedure.[22] When $a$ or $\sigma$, or both, are functions of time, a penalty function is often added to the goodness-of-fit measure so that the functions are

[22] For a good description of this procedure, see W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes: The Art of Scientific Computing*, 3rd edn. Cambridge University Press, 2007.

"well behaved". In the example just mentioned, where $\sigma$ is a step function, an appropriate objective function is

$$\sum_{i=1}^{n}(U_i - V_i)^2 + \sum_{i=1}^{n} w_{1,i}(\sigma_i - \sigma_{i-1})^2 + \sum_{i=1}^{n-1} w_{2,i}(\sigma_{i-1} + \sigma_{i+1} - 2\sigma_i)^2$$

The second term provides a penalty for large changes in $\sigma$ between one step and the next. The third term provides a penalty for high curvature in $\sigma$. Appropriate values for $w_{1,i}$ and $w_{2,i}$ are based on experimentation and are chosen to provide a reasonable level of smoothness in the $\sigma$ function.

The calibrating instruments chosen should be as similar as possible to the instrument being valued. Suppose, for example, that the model is to be used to value a Bermudan-style swap option that lasts 10 years and can be exercised on any payment date between year 5 and year 9 into a swap maturing 10 years from today. The most relevant calibrating instruments are $5 \times 5$, $6 \times 4$, $7 \times 3$, $8 \times 2$, and $9 \times 1$ European swap options. (An $n \times m$ European swap option is an $n$-year option to enter into a swap lasting for $m$ years beyond the maturity of the option.)

The advantage of making $a$ or $\sigma$, or both, functions of time is that the models can be fitted more precisely to the prices of instruments that trade actively in the market. The disadvantage is that the volatility structure becomes nonstationary. The volatility term structure given by the model in the future is liable to be quite different from that existing in the market today.[23]

A somewhat different approach to calibration is to use all available calibrating instruments to calculate "global-best-fit" $a$ and $\sigma$ parameters. The parameter $a$ is held fixed at its best-fit value. The model can then be used in the same way as Black–Scholes–Merton. There is a one-to-one relationship between options prices and the $\sigma$ parameter. The model can be used to convert tables such as Tables 29.1 and 29.2 into tables of implied $\sigma$'s.[24] These tables can be used to assess the $\sigma$ most appropriate for pricing the instrument under consideration.

## 31.9 HEDGING USING A ONE-FACTOR MODEL

Section 29.5 outlined some general approaches to hedging a portfolio of interest rate derivatives. These approaches can be used with the term structure models in this chapter. The calculation of deltas, gammas, and vegas involves making small changes to either the zero curve or the volatility environment and recomputing the value of the portfolio.

Note that, although one factor is often assumed when pricing interest rate derivatives, it is not appropriate to assume only one factor when hedging. For example, the deltas calculated should allow for many different movements in the yield curve, not just those that are possible under the model chosen. The practice of taking account of changes that

[23] For a discussion of the implementation of a model where $a$ and $\sigma$ are functions of time, see Technical Note 16 at www.rotman.utoronto.ca/~hull/TechnicalNotes.

[24] Note that in a term structure model the implied $\sigma$'s are not the same as the implied volatilities calculated from Black's model in Tables 29.1 and 29.2. The procedure for computing implied $\sigma$'s is as follows. The Black volatilities are converted to prices using Black's model. An iterative procedure is then used to imply the $\sigma$ parameter in the term structure model from the price.

cannot happen under the model considered, as well as those that can, is known as *outside model hedging* and is standard practice for traders.[25] The reality is that relatively simple one-factor models if used carefully usually give reasonable prices for instruments, but good hedging procedures must explicitly or implicitly assume many factors.

## SUMMARY

The traditional models of the term structure used in finance are known as equilibrium models. These are useful for understanding potential relationships between variables in the economy, but have the disadvantage that the initial term structure is an output from the model rather than an input to it. When valuing derivatives, it is important that the model used be consistent with the initial term structure observed in the market. No-arbitrage models are designed to have this property. They take the initial term structure as given and define how it can evolve.

This chapter has provided a description of a number of one-factor no-arbitrage models of the short rate. These are robust and can be used in conjunction with any set of initial zero rates. The simplest model is the Ho–Lee model. This has the advantage that it is analytically tractable. Its chief disadvantage is that it implies that all rates are equally variable at all times. The Hull–White model is a version of the Ho–Lee model that includes mean reversion. It allows a richer description of the volatility environment while preserving its analytic tractability. Lognormal one-factor models avoid the possibility of negative interest rates, but have no analytic tractability.

## FURTHER READING

### Equilibrium Models

Ahmad, R., and P. Wilmott, "The Market Price of Interest Rate Risk: Measuring and Modelling Fear and Greed in the Fixed Income Markets," *Wilmott*, January 2007: 64–70.

Cox, J. C., J. E. Ingersoll, and S. A. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53 (1985): 385–407.

Longstaff, F. A. and E. S. Schwartz, "Interest Rate Volatility and the Term Structure: A Two Factor General Equilibrium Model," *Journal of Finance*, 47, 4 (September 1992): 1259–82.

Vasicek, O. A.,"An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5 (1977): 177–88.

### No-Arbitrage Models

Black, F., E. Derman, and W. Toy, "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Prices," *Financial Analysts Journal*, January/February 1990: 33–39.

Black, F., and P. Karasinski, "Bond and Option Pricing When Short Rates Are Lognormal," *Financial Analysts Journal*, July/August (1991): 52–59.

Brigo, D., and F. Mercurio, *Interest Rate Models: Theory and Practice*, 2nd edn. New York: Springer, 2006.

Ho, T. S. Y., and S.-B. Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 41 (December 1986): 1011–29.

[25] A simple example of outside model hedging is in the way that the Black–Scholes–Merton model is used. The Black–Scholes–Merton model assumes that volatility is constant—but traders regularly calculate vega and hedge against volatility changes.

Hull, J., and A. White, "Bond Option Pricing Based on a Model for the Evolution of Bond Prices," *Advances in Futures and Options Research*, 6 (1993): 1–13.

Hull, J., and A. White, "Pricing Interest Rate Derivative Securities," *The Review of Financial Studies*, 3, 4 (1990): 573–92.

Hull, J., and A. White, "Using Hull–White Interest Rate Trees," *Journal of Derivatives*, Spring (1996): 26–36.

Rebonato, R., *Interest Rate Option Models*. Chichester: Wiley, 1998.

## Practice Questions (Answers in Solutions Manual)

31.1. What is the difference between an equilibrium model and a no-arbitrage model?

31.2. Suppose that the short rate is currently 4% and its standard deviation is 1% per annum. What happens to the standard deviation when the short rate increases to 8% in (a) Vasicek's model; (b) Rendleman and Bartter's model; and (c) the Cox, Ingersoll, and Ross model?

31.3. If a stock price were mean reverting or followed a path-dependent process there would be market inefficiency. Why is there not a market inefficiency when the short-term interest rate does so?

31.4. Explain the difference between a one-factor and a two-factor interest rate model.

31.5. Can the approach described in Section 31.4 for decomposing an option on a coupon-bearing bond into a portfolio of options on zero-coupon bonds be used in conjunction with a two-factor model? Explain your answer.

31.6. Suppose that $a = 0.1$ and $b = 0.1$ in both the Vasicek and the Cox, Ingersoll, Ross model. In both models, the initial short rate is 10% and the initial standard deviation of the short-rate change in a short time $\Delta t$ is $0.02\sqrt{\Delta t}$. Compare the prices given by the models for a zero-coupon bond that matures in year 10.

31.7. Suppose that $a = 0.1$, $b = 0.08$, and $\sigma = 0.015$ in Vasicek's model, with the initial value of the short rate being 5%. Calculate the price of a 1-year European call option on a zero-coupon bond with a principal of \$100 that matures in 3 years when the strike price is \$87.

31.8. Repeat Problem 31.7 valuing a European put option with a strike of \$87. What is the put–call parity relationship between the prices of European call and put options? Show that the put and call option prices satisfy put–call parity in this case.

31.9. Suppose that $a = 0.05$, $b = 0.08$, and $\sigma = 0.015$ in Vasicek's model with the initial short-term interest rate being 6%. Calculate the price of a 2.1-year European call option on a bond that will mature in 3 years. Suppose that the bond pays a coupon of 5% semiannually. The principal of the bond is 100 and the strike price of the option is 99. The strike price is the cash price (not the quoted price) that will be paid for the bond.

31.10. Use the answer to Problem 31.9 and put–call parity arguments to calculate the price of a put option that has the same terms as the call option in Problem 31.9.

31.11. In the Hull–White model, $a = 0.08$ and $\sigma = 0.01$. Calculate the price of a 1-year European call option on a zero-coupon bond that will mature in 5 years when the term structure is flat at 10%, the principal of the bond is \$100, and the strike price is \$68.

31.12. Suppose that $a = 0.05$ and $\sigma = 0.015$ in the Hull–White model with the initial term structure being flat at 6% with semiannual compounding. Calculate the price of a 2.1-year European call option on a bond that will mature in 3 years. Suppose that the bond pays a coupon of 5% per annum semiannually. The principal of the bond is 100 and the strike price of the option is 99. The strike price is the cash price (not the quoted price) that will be paid for the bond.

31.13. Observations spaced at intervals $\Delta t$ are taken on the short rate. The $i$th observation is $r_i$ $(0 \leqslant i \leqslant m)$. Show that the maximum likelihood estimates of $a$, $b$, and $\sigma$ in Vasicek's model are given by maximizing

$$\sum_{i=1}^{m}\left(-\ln(\sigma^2\Delta t) - \frac{[r_i - r_{i-1} - a(b - r_{i-1})\Delta t]^2}{\sigma^2\Delta t}\right)$$

What is the corresponding result for the CIR model?

31.14. Suppose $a = 0.05$, $\sigma = 0.015$, and the term structure is flat at 10%. Construct a trinomial tree for the Hull–White model where there are two time steps, each 1 year in length.

31.15. Calculate the price of a 2-year zero-coupon bond from the tree in Figure 31.6.

31.16. Calculate the price of a 2-year zero-coupon bond from the tree in Figure 31.9 and verify that it agrees with the initial term structure.

31.17. Calculate the price of an 18-month zero-coupon bond from the tree in Figure 31.10 and verify that it agrees with the initial term structure.

31.18. What does the calibration of a one-factor term structure model involve?

31.19. Use the DerivaGem software to value $1 \times 4$, $2 \times 3$, $3 \times 2$, and $4 \times 1$ European swap options to receive fixed and pay floating. Assume that the 1-, 2-, 3-, 4-, and 5-year interest rates are 6%, 5.5%, 6%, 6.5%, and 7%, respectively. The payment frequency on the swap is semiannual and the fixed rate is 6% per annum with semiannual compounding. Use the Hull–White model with $a = 3\%$ and $\sigma = 1\%$. Calculate the volatility implied by Black's model for each option.

31.20. Prove equations (31.25), (31.26), and (31.27).

31.21. (a) What is the second partial derivative of $P(t, T)$ with respect to $r$ in the Vasicek and CIR models.
(b) In Section 31.2, $\hat{D}$ is presented as an alternative to the standard duration measure $D$. What is a similar alternative $\hat{C}$ to the convexity measure in Section 4.9?
(c) What is $\hat{C}$ for $P(t, T)$? How would you calculate $\hat{C}$ for a coupon-bearing bond?
(d) Give a Taylor series expansion for $\Delta P(t, T)$ in terms of $\Delta r$ and $(\Delta r)^2$ for Vasicek and CIR.

31.22. Suppose that short rate $r$ is 4% and its real-world process is $dr = 0.1[0.05 - r]\,dt + 0.01\,dz$, while the risk-neutral process is $dr = 0.1[0.11 - r]\,dt + 0.01\,dz$.
(a) What is the market price of interest rate risk?
(b) What is the expected return and volatility for a 5-year zero-coupon bond in the risk-neutral world?
(c) What is the expected return and volatility for the 5-year zero-coupon bond in the real world?

## Further Questions

31.23. Construct a trinomial tree for the Ho–Lee model where $\sigma = 0.02$. Suppose that the the initial zero-coupon interest rate for a maturities of 0.5, 1.0, and 1.5 years are 7.5%, 8%, and 8.5%. Use two time steps, each 6 months long. Calculate the value of a zero-coupon bond with a face value of \$100 and a remaining life of 6 months at the ends of the final nodes of the tree. Use the tree to value a 1-year European put option with a strike price of 95 on the bond. Compare the price given by your tree with the analytic price given by DerivaGem.

31.24. A trader wishes to compute the price of a 1-year American call option on a 5-year bond with a face value of 100. The bond pays a coupon of 6% semiannually and the (quoted) strike price of the option is \$100. The continuously compounded zero rates for maturities of 6 months, 1 year, 2 years, 3 years, 4 years, and 5 years are 4.5%, 5%, 5.5%, 5.8%, 6.1%, and 6.3%. The best-fit reversion rate for either the normal or the lognormal model has been estimated as 5%.

A 1-year European call option with a (quoted) strike price of 100 on the bond is actively traded. Its market price is \$0.50. The trader decides to use this option for calibration. Use the DerivaGem software with 10 time steps to answer the following questions:

(a) Assuming a normal model, imply the $\sigma$ parameter from the price of the European option.
(b) Use the $\sigma$ parameter to calculate the price of the option when it is American.
(c) Repeat (a) and (b) for the lognormal model. Show that the model used does not significantly affect the price obtained providing it is calibrated to the known European price.
(d) Display the tree for the normal model and calculate the probability of a negative interest rate occurring.
(e) Display the tree for the lognormal model and verify that the option price is correctly calculated at the node where, with the notation of Section 31.7, $i = 9$ and $j = -1$.

31.25. Use the DerivaGem software to value $1 \times 4$, $2 \times 3$, $3 \times 2$, and $4 \times 1$ European swap options to receive floating and pay fixed. Assume that the 1-, 2-, 3-, 3-, and 5-year interest rates are 3%, 3.5%, 3.8%, 4.0%, and 4.1%, respectively. The payment frequency on the swap is semiannual and the fixed rate is 4% per annum with semiannual compounding. Use the lognormal model with $a = 5\%$, $\sigma = 15\%$, and 50 time steps. Calculate the volatility implied by Black's model for each option.

31.26. Verify that the DerivaGem software gives Figure 31.11 for the example considered. Use the software to calculate the price of the American bond option for the lognormal and normal models when the strike price is 95, 100, and 105. In the case of the normal model, assume that $a = 5\%$ and $\sigma = 1\%$. Discuss the results in the context of the heavy-tails arguments of Chapter 20.

31.27. Modify Sample Application G in the DerivaGem Application Builder software to test the convergence of the price of the trinomial tree when it is used to price a 2-year call option on a 5-year bond with a face value of 100. Suppose that the strike price (quoted) is 100, the coupon rate is 7% with coupons being paid twice a year. Assume that the zero curve is as in Table 31.2. Compare results for the following cases:

(a) Option is European; normal model with $\sigma = 0.01$ and $a = 0.05$
(b) Option is European; lognormal model with $\sigma = 0.15$ and $a = 0.05$

(c) Option is American; normal model with $\sigma = 0.01$ and $a = 0.05$
(d) Option is American; lognormal model with $\sigma = 0.15$ and $a = 0.05$.

31.28. Suppose that the (CIR) process for short-rate movement in the (traditional) risk-neutral world is

$$dr = a(b - r)\,dt + \sigma\sqrt{r}\,dz$$

and the market price of interest rate risk is $\lambda$.

(a) What is the real world process for $r$?
(b) What is the expected return and volatility for a 10-year bond in the risk-neutral world?
(c) What is the expected return and volatility from a 10-year bond in the real world?