REGULARIDAD DE SOLUCIONES DE ECUACIONES ELÍPTICAS NO LINEALES.

$$\begin{cases} -\Delta u = f(x,u) & \text{EN } \Omega \subseteq \mathbb{R}^n \\ u = 0 & \text{EN } \partial\Omega \end{cases}$$

TEOREMA

SI $f \in \mathcal{F}_p$, $1 < p < \frac{n+2}{n-2}$ ENTONCES LA SOLUCIÓN DÉBIL $u \in H_0^1(\Omega)$ ES TAMBIÉN CLÁSICA: $u \in C^{2,\alpha}(\Omega)$
$(\alpha > 0)$

DEMOSTRACIÓN

$$f \in \mathcal{F}_p \Rightarrow |f(x,u)| \leq a_1(x) + a_2 |u|^p$$

$a_1(x)$ HÖLDER CLASE $C^{0,\alpha}(\Omega)$

a_2 CONSTANTE, $1 < p < \frac{n+2}{n-2}$

PARA SIMPLIFICAR: $a_1 \equiv 0$, $a_2 = 1$, $n \geq 3$.

EJERCICIO GENERALIZACIÓN PARA

(i) $n = 2$

(ii) $a_1(x) \not\equiv 0$.

ARGUMENTO DE "BOOTSTRAP"

$$-\Delta u = f(u) \quad \text{EN } \Omega \subseteq \mathbb{R}^n, \ n \geq 3$$

$$u = 0 \quad \text{EN } \partial\Omega$$

(1) SOLUCIÓN DÉBIL $u \in H_0^1(\Omega) \Longrightarrow$
$u \in L^{2n/n-2}$ (SOBOLEV)

(2) $$\int_\Omega |f(u)|^\beta \leq \int_\Omega |u|^{p\beta} < \infty$$

CUANDO $$p\beta < \frac{2n}{n-2}$$

$$p < \frac{n+2}{n-2}$$

$$p\beta < \frac{n+2}{n-2}\beta < \frac{2n}{n-2}$$

QUEREMOS $\beta > 1$.

BASTA

$$\beta < \frac{2n}{n+2}$$

CON $n \geq 3 \Rightarrow \beta < \frac{6}{5}$

$\Rightarrow$ EXISTE UN $\beta > 1$ TAL QUE

$$f(u) \in L^\beta(\Omega)$$

$\Longrightarrow$ VOLVIENDO A LA ECUACIÓN

$$-\Delta u = f(u) \in L^\beta(\Omega)$$

USANDO UNO DE LOS TEOREMAS ANTERIORES

$$\Delta u \in L^\beta(\Omega) \Longrightarrow u \in H^{2,\beta}(\Omega)$$

SI $2\beta > n \implies$ $-\Delta u = f(u) \in C^{0,\alpha}$
CIERTO $\alpha > 0$

$u \in H^{2,\beta}(\Omega)$ CON $\beta > \frac{n}{2}$

$\implies u \in C^{0,\alpha}(\Omega)$

(SOBOLEV)

$\implies -\Delta u \in C^{0,\alpha}(\Omega)$

LAS COTAS DE SCHAUDER

$\implies u \in C^{2,\alpha}(\Omega)$

(3) SI $2\beta < n$ REPETIMOS PASOS (1) Y (2)

USANDO SOBOLEV:

$$H^{2,\beta}(\Omega) \hookrightarrow L^{n\beta/n-2\beta}(\Omega)$$

$$\implies u \in L^{n\beta/n-2\beta}(\Omega)$$

COMENZAMOS CON $u \in L^{2n/n-2}(\Omega)$

AFIRMACIÓN:

$$\frac{n\beta}{n-2\beta} > \frac{2n}{n-2} \quad \text{SI} \quad \beta > 1, \quad n \geq 3$$

$$\implies \int |f(u)|^{\beta_1} \leq \int |u|^{p\beta_1}$$

CON $\beta_1 > \beta$

$\Longrightarrow u \in H^{2,\beta_1}(\Omega)$, $\beta_1 > \beta$

SI $2\beta_1 > n \Longrightarrow$ TERMINAMOS

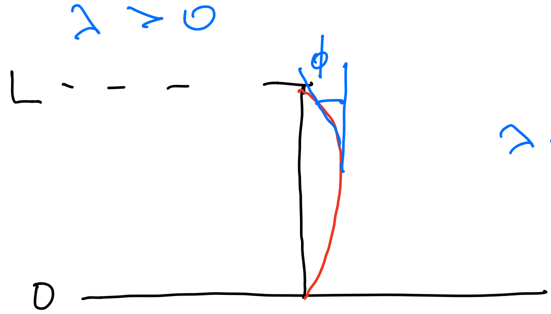
SI NO REPETIMOS

DOS EJEMPLOS DE APLICACIÓN DEL TEOREMA DE BIFURCACIONES.

(I)

$$\begin{cases} \phi'' + \lambda \sin\phi = 0, & x \in (0,L) \\ \phi'(0) = \phi'(L) = 0 \end{cases}$$

$\lambda > 0$



$\lambda \sim$ FUERZA APLICADA
$\lambda > 0$

$X = \{ \phi \in C^2([0,L]) \mid \phi'(0) = \phi'(L) = 0 \}$

$Y = C([0,L])$

$S(\lambda, \phi) = \phi'' + \lambda \sin\phi$

OBSERVAMOS $S(\lambda, 0) = 0$, $\forall \lambda > 0$

$D_\phi S(\lambda, 0)[h] = h'' + \lambda h = \mathcal{L}$

$$D_\phi S(\lambda,\phi)[h] = h'' + \lambda \cos\phi \, h$$

$$\phi = 0 \Longrightarrow \cos\phi = 1$$

$$\operatorname{Ker} \mathcal{L} \doteq \{ h \mid \mathcal{L}h = 0,\ h \in X \}$$

$$\operatorname{Ker} \mathcal{L} : \begin{cases} h'' + \lambda h = 0 \\ h'(0) = h'(L) = 0 \end{cases}$$

$$\Longrightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}$$

$$h_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

TODOS VALORES PROPIOS SON SIMPLES.

PARA APLICAR EL TEOREMA ESCRIBIMOS

$$S(\lambda,\phi) = 0 \Longleftrightarrow$$

$$\phi - \lambda K(\phi) - \lambda K(\sin\phi - \phi) = 0$$

$$\phi \in X$$

K ES EL OPERADOR DE GREEN DE $\dfrac{d^2}{dx^2}$ EN X:

$$\phi = K(f) \quad \text{si}$$

$$-\phi'' = f$$
$$\phi'(0) = \phi'(L) = 0$$

$$S(\lambda, \phi) = 0 \iff$$

$$\phi = \lambda K(\sin\phi) = \lambda K(\sin\phi - \phi + \phi)$$

$$= \lambda K(\phi) + \lambda K(\sin\phi - \phi)$$

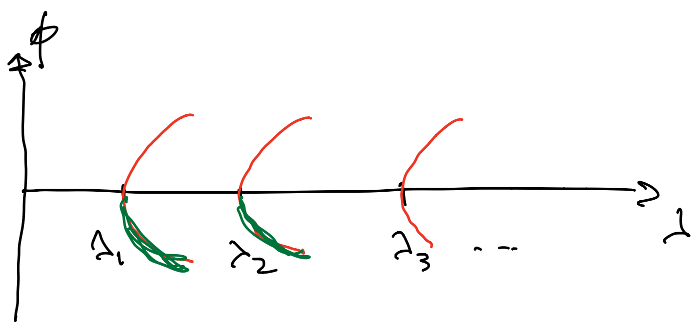
$\Rightarrow$

$$\phi - \lambda K(\phi) - \lambda K(\sin\phi - \phi)$$

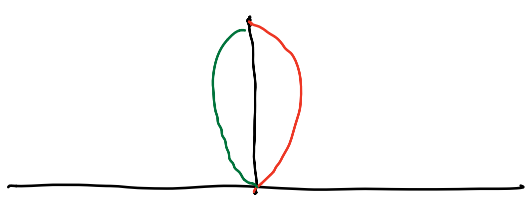
(i) K ES COMPACTO (EJERCICIO)

(ii) $T(\phi) = K(\sin\phi - \phi)$ ~~SA~~TISFACE

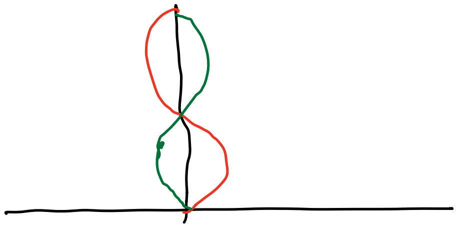
$$T(0) = 0, \quad DT(0) = 0$$



$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$



$\lambda = \lambda_1$



$\lambda = \lambda_2$

EJEMPLO: BIFURCACIONES DE SOLUCIONES PERIODICAS

SEAN $\delta > 0$, $A, B \in C^2((-\delta,\delta) \times \overset{t}{\mathbb{R}} \times (-\delta,\delta))$ PERIODICAS DE PERIODO 2π EN t

CONSIDERAMOS:

$$\ddot{w}(t) + \lambda w(t) + w^2(t) A(w(t), t, \lambda) + \lambda^2 w(t) B(w(t), t, \lambda) = 0$$

$$w(0) = w(2\pi)$$
$$w \in C^1([0, 2\pi])$$

$S(\lambda, w)$ = EL LADO IZQUIERDO DE LA ECUACIÓN

$$X = \{ u \in C^1[0, 2\pi] \mid u(0) = u(2\pi) \}$$

$$Y = \{ u \in C[0, 2\pi] \mid u(0) = u(2\pi) \}$$

$$S : U \to V, \quad U \subseteq X, \ V \subseteq Y$$

$$S \in C^2(U; Y)$$

$$U = (-\delta, \delta) \times \{ \sup_t |u(t)| < \delta \}$$

$$\subseteq \mathbb{R} \times X$$

OBVIO: $S(\lambda, 0) = 0 \quad , \quad \forall \lambda \in (-\delta, \delta)$

$$D_u S(\lambda, u)[h] = \dot{h} + \lambda h$$

$$+ 2 h u A(u, t, \lambda) + u^2 \frac{\partial A}{\partial u}(u, t, \lambda) h$$

$$+ \lambda^2 h B(u, t, \lambda) + \lambda^2 u \frac{\partial B}{\partial u}(u, t, \lambda) h$$

$$D_u S(0,0) = \dot{h} = Lh$$

$$\boxed{D_u S(\lambda, 0) = \dot{h} + \lambda h}$$

$$Lh = \dot{h} = 0$$

$$h(0) = h(2\pi) \implies h = 1 \in \operatorname{Ker} L$$

$h = 1$ CORRESPONDE AL VALOR PROPIO SIMPLE DE $L = 0$

$\lambda = 0$ ES SOSPECHOSO DE SER UN PUNTO DE BIFURACIÓN

$$\operatorname{Ker} D_u S(0,0) = \{ u \in X \mid u = \text{const} \}$$

$$\operatorname{Ran} D_u S(0,0) = \{ f \in Y \mid \int_0^{2\pi} f = 0 \}$$

$\Rightarrow$ VERIFICAR QUE $D_u S(0,0)$ ES UN OPERADOR DE FREDHOLM DE INDICE 0, EN PARTICULAR SE CUMPLEN (1') Y (2') DE TEOREMA DE BIFURCACIÓN (EJERCICIO)

BASTA VERIFICAR LA CONDICIÓN DE TRANSVERSABILIDAD:

$\xi \in \operatorname{Ker} D_u S(0,0)$
$\xi = \text{const}$

$$D^2_{\lambda u} S(0,0)[\lambda, \xi] \notin \operatorname{Ran} D_u S(0,0)$$

$$D^2_{\lambda u} S(0,0)[\lambda, \xi] = \xi = \text{const} \notin \operatorname{Ran} D_u S(0,0)$$

$$\int_0^{2\pi} \xi = 2\pi\xi \neq 0$$

$\Rightarrow$ EXISTE UNA CURVA DE SOLUCIONES

$s \in (-\varepsilon, \varepsilon)$

$\lambda = \lambda(s)$, $\lambda(0) = 0$

$w = w(t;s) = s\xi + u(t;s)$
$u(t,0) = 0$

$$S(\lambda(s), w(t,s)) = 0, \quad \forall s \in (-\varepsilon, \varepsilon)$$

PUEDE PASAR $u(t,s) \equiv 0$

$\Rightarrow$ $w(t;s) = s\xi = \text{CONST.}$

DEMOSTRAR QUE SI

$$\partial_t A(0,t,0) \neq 0$$

$\Rightarrow$ LAS SOLUCIONES $w(t;s)$ SON NO TRIVIALES (NO SON CONSTANTES)