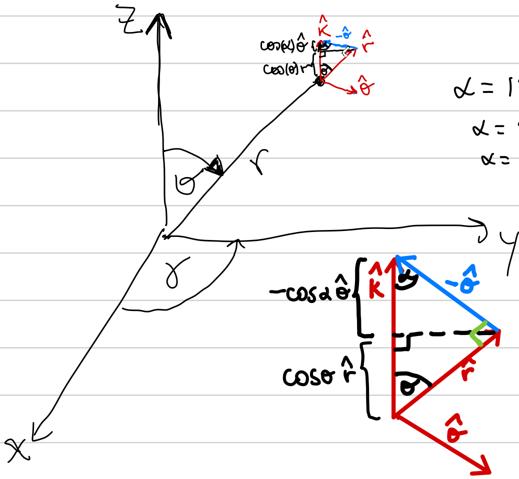
Auxiliar 3: Coordenadas curvilíneas

P1. a) Si $\hat{k}$ es el vector unitario según el eje Z en el plano cartesiano, θ el ángulo cenital y $\hat{r}$ el vector unitario radial en esféricas, demuestre que

$$\hat{k} = \cos(\theta)\,\hat{r} - \sin(\theta)\,\hat{\theta}$$



$\alpha = 180 - 90 - \theta$

$\alpha = 90 - \theta$

$\alpha = \frac{\pi}{2} - \theta$

$\cos\alpha = \cos\left(\frac{\pi}{2} - \theta\right)$

$= \text{sen}(\theta)$

$\hat{k} = \cos(\theta)\,\hat{r} - \text{sen}(\theta)\,\hat{\theta}$ //

b) Considere para un α real, el campo escalar $f : \mathbb{R}^3 \to \mathbb{R}$ definido por

$$f = \frac{\alpha}{4\pi}\frac{\hat{k}\cdot\hat{r}}{r^2}$$

Calcule $F = -\nabla f$, expresado en la base de coordenadas esféricas.

$$f = \frac{\alpha}{4\pi}\,\frac{(\cos(\theta)\,\hat{r} - \cancel{\text{sen}(\theta)\,\hat{\theta}})\cdot\hat{r}}{r^2}$$

$\hat{r}\cdot\hat{r} = 1$

$$f = \frac{\alpha\cos(\theta)}{4\pi r^2} \quad \Big\} \text{ f en coordenadas esfericas}$$

$$\nabla f = \frac{1}{h_u}\frac{df}{du}\hat{u} + \frac{1}{h_v}\frac{df}{dv}\hat{v} + \frac{1}{h_w}\frac{df}{dw}\hat{w}$$

$$\boxed{\begin{array}{l} u = r \\ v = \theta \\ w = \varphi \end{array}} \qquad \boxed{\begin{array}{l} h_r = 1 \\ h_\theta = r \\ h_\varphi = r\,\mathrm{sen}\,\theta \end{array}}$$

$$\nabla f = \frac{df}{dr}\hat{r} + \frac{1}{r}\frac{df}{d\theta}\hat{\theta} = \frac{\alpha\cos(\theta)}{4\pi}\frac{d}{dr}\left(\frac{1}{r^2}\right)\hat{r}$$

$$+ \frac{1}{r}\cdot\frac{\alpha}{4\pi r^2}\cdot\frac{d}{d\theta}\left(\cos(\theta)\right)\hat{\theta}$$

$$= \frac{-2\alpha\cos(\theta)}{4\pi r^3}\hat{r} + \frac{(-\mathrm{sen}(\theta))\cdot\alpha\cdot\hat{\theta}}{4\pi r^3}$$

$$= \frac{-\alpha}{4\pi r^3}\left(2\cos(\theta)\,\hat{r} + \mathrm{sen}(\theta)\,\hat{\theta}\right)$$

$$F = -\nabla f = \frac{\alpha}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \mathrm{sen}\,\theta\cdot\hat{\theta}\right) //.$$

c) Considere ahora el campo vectorial $A: \mathbb{R}^3 \to \mathbb{R}^3$ definido por

$$A = \frac{\alpha}{4\pi}\frac{\hat{k}\times\hat{r}}{r^2}$$

Calcule su rotor $\nabla \times A$.

$$\hat{k} = \cos\theta\,\hat{r} - \operatorname{sen}\theta\,\hat{\theta}$$

$$A = \frac{\alpha}{4\pi}\frac{(\cos\theta\,\hat{r} - \operatorname{sen}\theta\,\hat{\theta})\times\hat{r}}{r^2}$$

$$= \frac{\alpha}{4\pi}\cdot\frac{1}{r^2}\left(\cancelto{0}{\cos\theta\,\hat{r}\times\hat{r}} - \operatorname{sen}\theta\,\hat{\theta}\times\hat{r}\right)$$

$$\cos\theta\,\hat{r}\times\hat{r} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\varphi} \\ \cos\theta & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\operatorname{sen}\theta\,\hat{\theta}\times\hat{r} \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\varphi} \\ 0 & \operatorname{sen}\theta & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\operatorname{sen}\theta\,\hat{\varphi}$$

$$A = \frac{\alpha(\operatorname{sen}\theta\,\hat{\varphi})}{4\pi r^2}$$

$$\operatorname{rot}\vec{A} = \frac{1}{r^2\operatorname{sen}\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\operatorname{sen}\theta\,\hat{\varphi} \\ \frac{d}{dr} & \frac{d}{d\theta} & \frac{d}{d\varphi} \\ 0 & 0 & \underbrace{r\operatorname{sen}\theta\cdot\frac{\alpha\operatorname{sen}\theta}{4\pi r^2}}_{F_\varphi\cdot h_\varphi} \end{vmatrix}$$

formula rotor coordenadas ortogonales

$$= \frac{\frac{d}{d\theta}\left(\frac{\cancel{r}\alpha\operatorname{sen}^2\theta}{4\pi r^{\cancel{2}}}\right)\hat{r} - \frac{d}{dr}\left(\frac{r\alpha\operatorname{sen}^2\theta}{4\pi r^2}\right)\cdot r\cdot\hat{\theta}}{r^2\operatorname{sen}\theta}$$

$$= \frac{\frac{\alpha}{4\pi r} \cdot \frac{d}{d\theta}(\text{sen}^2\theta)\hat{r} - \frac{\alpha\,\text{sen}^2\theta}{4\pi} \cdot \frac{d}{dr}\left[\frac{1}{r}\right]\hat{\theta} \cdot r}{r^2\,\text{sen}\theta}$$

$$= \frac{2\,\text{sen}\theta \cdot \frac{d}{d\theta}(\text{sen}\theta) \cdot \frac{\alpha}{4\pi r}\hat{r} - \left(\frac{-\frac{d}{dr}[r]}{r^2} \cdot \frac{\alpha\,\text{sen}^2\theta}{4\pi}\right) \cdot r \cdot \hat{\theta}}{r^2\,\text{sen}\theta}$$

$$= \frac{\frac{\alpha \cos\theta\,\cancel{\text{sen}\theta}}{2\pi r}\hat{r} + \frac{\alpha\,\text{sen}^{\cancel{2}}\theta}{4\pi r^{\cancel{2}}} \cdot \cancel{r} \cdot \hat{\theta}}{r^2\,\cancel{\text{sen}\theta}}$$

$$= \frac{\alpha}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \text{sen}\theta\,\hat{\theta}\right) = \text{rot}\vec{A}.$$

d) De acuerdo a la teoría de Yukawa para las fuerzas nucleares, la fuerza de atracción entre un neutrón y un protón tiene como potencial, en coordenadas esféricas:

$$U(r) = -K\frac{e^{-\alpha r}}{r}$$

Donde $K, \alpha > 0$. Encuentre una fuerza $\vec{F}$ tal que, en $\mathbb{R}^3 \setminus \{0\}$:

$$\vec{F} = -\nabla U$$

$$\text{grad } F = \frac{1}{h_1}\frac{df}{du_1}\hat{u}_1 + \frac{1}{h_2}\frac{df}{du_2}\hat{u}_2 + \frac{1}{h_3}\frac{df}{du_3}\hat{u}_3$$

$$\boxed{\begin{matrix} u_1 = r \\ u_2 = \theta \\ u_3 = \varphi \end{matrix}} \quad \left.\boxed{\begin{matrix} h_r = 1 \\ h_\theta = r\,\text{sen}\,\varphi \\ h_\varphi = r \end{matrix}}\right\} \text{factores de escala}$$

$$\underset{\text{grad}}{\nabla U} = \frac{1}{h_r}\cdot\frac{dU}{dr}\hat{r}$$

$$= \frac{1}{1}\left(-K\frac{d}{dr}\left(\frac{e^{-\alpha r}}{r}\right)\right)\hat{r}$$

$$= \left(-K\left[\frac{\frac{d}{dr}[e^{-\alpha r}]\cdot r - e^{-\alpha r}\cdot \cancel{\frac{d}{dr}[r]}^{1}}{r^2}\right]\right)\hat{r}$$

$$= -\frac{K}{r^2}\left[e^{-\alpha r}\cdot\frac{d}{dr}[-\alpha r]\cdot r - e^{-\alpha r}\right]\hat{r}$$

$$= -\frac{K}{r^2}\left[e^{-\alpha r}\cdot -\alpha\cdot r - e^{-\alpha r}\right]\hat{r}$$

$$\nabla U = -\frac{K e^{-\alpha r}}{r^2}\left[-\alpha r - 1\right]\hat{r} = \frac{K e^{-\alpha r}}{r^2}\left[\alpha r + 1\right]\hat{r}$$

$$\vec{F} = -\nabla U = -\frac{K e^{-\alpha r}}{r^2}(\alpha r + 1)\hat{r} \ ///$$

e) Pruebe que $\Delta U = \alpha^2 U$ en $\mathbb{R}^3 \setminus \{0\}$.

$$\underset{\text{laplaciano}}{\Delta U} = \operatorname{div}(\underbrace{\nabla U}_{\text{gradiente}})$$

$$\Delta U = \operatorname{div}(\nabla U) = \frac{1}{h_1 h_2 h_3} \cdot \left[\frac{d}{dU_1}(\nabla U_r \cdot h_2 \cdot h_3) + \frac{d}{dU_2}(\nabla U_\theta h_1 h_3) + \frac{d}{dU_3}(\nabla U_\varphi h_1 h_2) \right]$$

$$= \frac{1}{1 \cdot r \cdot r \operatorname{sen}\varphi} \left[\frac{d}{dr}(r^2 \operatorname{sen}\varphi \cdot \nabla U_r) + \frac{d}{d\theta}(r \operatorname{sen}\varphi \cdot \nabla U_\theta) + \frac{d}{d\varphi}(r \cdot \nabla U_\varphi) \right]$$

U solo depende de r, luego:

$$\Delta U = \frac{1}{r^2 \operatorname{sen}\varphi} \left(\frac{d}{dr}(r^2 \operatorname{sen}\varphi \cdot \nabla U_r) \right) = \frac{1}{r^2 \operatorname{sen}\varphi} \left(\frac{d}{dr}\left(\cancel{r^2} \operatorname{sen}\varphi \cdot \overbrace{\frac{K \cdot e^{-\alpha r}}{\cancel{r^2}}(\alpha r + 1)}^{d)}\right) \right)$$

$$= \frac{1}{r^2 \cancel{\operatorname{sen}\varphi}} K \cdot \cancel{\operatorname{sen}\varphi} \cdot \frac{d}{dr}\left(e^{-\alpha r}(\alpha r + 1)\right)$$

$$= \frac{K}{r^2} \left[\frac{d(e^{-\alpha r})}{dr} \cdot (\alpha r + 1) + e^{-\alpha r} \cdot \frac{d}{dr}(\alpha r + 1) \right]$$

$$= \frac{K}{r^2} \cdot \left[-\alpha e^{-\alpha r}(\alpha r + 1) + \alpha e^{-\alpha r} \right]$$

$$= \frac{K}{r^2} \cdot \left[-\alpha e^{-\alpha r} \cdot \alpha r \cancel{- \alpha e^{-\alpha r}} \cancel{+ \alpha e^{-\alpha r}} \right]$$

$$= \frac{K}{r^{\cancel{2}}} \cdot -\alpha^2 \cancel{r} e^{-\alpha r} = -\alpha^2 \frac{K e^{-\alpha r}}{r} = \alpha^2 U \,//.$$

P2. Para un valor $\alpha > 0$ dado, se definen las coordenadas elípticas como

$$\vec{r}(u, v, z) = (a\cosh(u)\cos(v)\,, a\,senh(u)\,\mathrm{sen}(v), z)$$

Demuestre que estas efectivamente definen un sistema ortogonal de coordenadas y calcule sus factores de escala.

$$h_u = \left\| \frac{d\vec{r}}{du} \right\|$$

$$\frac{d\vec{r}}{du}(u,v,z) = (a\,\mathrm{senh}(u)\cos(v),\ a\cosh(u)\,\mathrm{sen}(v),\ 0)$$

$$\left\|\frac{d\vec{r}}{du}\right\| = \sqrt{a^2\,\mathrm{senh}^2(u)\cos^2(v) + a^2\cosh^2(u)\,\mathrm{sen}^2(v)}$$

$$= a\sqrt{\mathrm{senh}^2(u)\cos^2(v) + \cosh^2(u)\,\mathrm{sen}^2(v)}$$

$$= a\sqrt{\mathrm{senh}^2(u)\left[1-\mathrm{sen}^2(v)\right] + \cosh^2(u)\,\mathrm{sen}^2(v)}$$

$$= a\sqrt{\mathrm{senh}^2(u) - \mathrm{senh}^2(u)\,\underline{\mathrm{sen}^2(v)} + \cosh^2(u)\,\underline{\mathrm{sen}^2(v)}}$$

$$= a\sqrt{\mathrm{senh}^2(u) + \mathrm{sen}^2(v)\underbrace{\left[\cosh^2(u) - \mathrm{senh}^2(u)\right]}_{1}}$$

$$h_u = a\sqrt{\mathrm{senh}^2(u) + \mathrm{sen}^2(v)}$$

$$\frac{d\vec{r}}{dv}(u,v,z) = (-a\cosh(u)\,\mathrm{sen}(v),\ a\,\mathrm{senh}(u)\cos(v), 0)$$

$$h_v = \left\|\frac{d\vec{r}}{dv}\right\| = a\sqrt{\cosh^2(u)\,\mathrm{sen}^2(v) + \mathrm{senh}^2(u)\cos^2(v)}$$

$$= a\sqrt{\mathrm{senh}^2(u) + \mathrm{sen}^2(v)}$$

$$h_z = \left\|\frac{d\vec{r}}{dz}\right\| = 1 \ //.$$

$$\hat{u} = \begin{bmatrix} f_1(u,v) \\ f_2(u,v) \\ 0 \end{bmatrix} \quad \hat{v} = \begin{bmatrix} -f_2(u,v) \\ f_1(u,v) \\ 0 \end{bmatrix} \quad \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\hat{u}\cdot\hat{v}\cdot\hat{z} = 0$

$\Rightarrow$ ortogonales

P3. Se dice que un campo vectorial tiene simetría axial en $\mathbb{R}^3$ si es de la forma

$$\vec{F}(x,y,z) = g(\rho)\hat{\rho}$$

donde $\rho = \sqrt{x^2+y^2}$, $\hat{\rho} = (x/\rho\ , y/\rho\ , 0)$, y g es alguna función de clase C^1.

a) Calcule $div\vec{F}$ para un campo con simetría axial. Exprese sus resultados en términos de ρ , $g(\rho)$, $g'(\rho)$.

b) Cuál es la forma más general de g para un campo de simetría axial, que tiene divergencia nula.

$$\vec{F} = g(\rho)\hat{\rho} \longrightarrow F_\rho = g(\rho) \qquad h_\rho = h_K = 1$$
$$h_\theta = \rho$$

$$\operatorname{div}\vec{F} = \frac{1}{\rho}\cdot\frac{d}{d\rho}(\rho\cdot g(\rho)) \qquad \frac{1}{h_\rho\cdot h_\theta\cdot h_K}\cdot\frac{d}{d\rho}(h_\theta\cdot h_K\cdot F_\rho)$$

$$= \frac{1}{\rho}\cdot\left[\cancelto{1}{\frac{d\rho}{d\rho}}\cdot g(\rho) + \rho\cdot\frac{d}{d\rho}(g(\rho))\right]$$

$$= \frac{1}{\rho}\left[g(\rho)+\rho\cdot g'(\rho)\right]$$

$$= \frac{g(\rho)}{\rho} + g'(\rho) \ //$$

b) $\operatorname{div}\vec{F} = 0 \longrightarrow \underbrace{\frac{1}{\rho}}_{\neq 0}\cdot\frac{d}{d\rho}(\rho\cdot g(\rho)) = 0$

$$\Longrightarrow \rho\cdot g(\rho) = C$$

Entonces $= \boxed{g(\rho) = \frac{C}{\rho}}$ //