P1 (a) Notamos que

$$\lambda^2 = \sqrt{x^2+y^2+z^2} + z \quad , \quad \mu^2 = \sqrt{x^2+y^2+z^2} - z$$

$$\Rightarrow \lambda^2 - \mu^2 = 2z \quad \Rightarrow \quad z = \frac{1}{2}(\lambda^2-\mu^2) \qquad (+0.5)$$

Además

$$\lambda^2 + \mu^2 = 2\sqrt{x^2+y^2+z^2}$$

Reemplazando z y despejando x^2+y^2, vemos que

$$x^2+y^2 = \frac{1}{4}(\lambda^2+\mu^2)^2 - \frac{1}{4}(\lambda^2-\mu^2)^2 = \lambda^2\mu^2 \qquad (+0.5)$$

Pero $y = x\,tg(\varphi)$. Luego

$$x^2(1+tg^2(\varphi)) = \lambda^2\mu^2 \quad \Rightarrow \quad x^2 = \lambda^2\mu^2\cos^2(\varphi) \quad \Rightarrow x = \lambda\mu\cos(\varphi)$$

$$y = x\,tg(\varphi) = \lambda\mu\,sen(\varphi)$$

$$\therefore \quad x = \lambda\mu\cos(\varphi) \quad , \quad y = \lambda\mu\,sen(\varphi) \quad , \quad z = \frac{1}{2}(\lambda^2-\mu^2) \qquad (+0.5)$$

(b) Considerando $\vec{r}(\lambda,\mu,\varphi) = (\lambda\mu\cos(\varphi), \lambda\mu\,sen(\varphi), \frac{1}{2}(\lambda^2-\mu^2))$, vemos que

$$\frac{\partial\vec{r}}{\partial\lambda} = (\mu\cos(\varphi), \mu\,sen(\varphi), \lambda)$$

$$\frac{\partial\vec{r}}{\partial\mu} = (\lambda\cos(\varphi), \lambda\,sen(\varphi), -\mu)$$

$$\frac{\partial\vec{r}}{\partial\varphi} = (-\lambda\mu\,sen(\varphi), \lambda\mu\cos(\varphi), 0)$$

Así, $h_\lambda = \left\|\frac{\partial\vec{r}}{\partial\lambda}\right\| = \sqrt{\lambda^2+\mu^2}$, $h_\mu = \left\|\frac{\partial\vec{r}}{\partial\mu}\right\| = \sqrt{\lambda^2+\mu^2}$, $h_\varphi = \left\|\frac{\partial\vec{r}}{\partial\varphi}\right\| = \lambda\mu$.

(+ 0.5)

Con esto, los vectores unitarios $\hat{\lambda}, \hat{\mu}, \hat{\varphi}$ resultan

$$\hat{\lambda} = \left(\frac{\mu}{\sqrt{\lambda^2+\mu^2}}\cos(\varphi), \frac{\mu}{\sqrt{\lambda^2+\mu^2}}\,sen(\varphi), \frac{\lambda}{\sqrt{\lambda^2+\mu^2}}\right)$$

$$\hat{\mu} = \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}}\cos(\varphi), \frac{\lambda}{\sqrt{\lambda^2+\mu^2}}\,sen(\varphi), \frac{-\mu}{\sqrt{\lambda^2+\mu^2}}\right)$$

$$\hat{\varphi} = \big(-\operatorname{sen}(\varphi),\ \cos(\varphi),\ 0\big). \qquad (+0.5)$$

Notamos finalmente que

$$\hat{\lambda} \times \hat{\mu} = \frac{1}{\lambda^2+\mu^2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \mu\cos(\varphi) & \mu\operatorname{sen}(\varphi) & \lambda \\ \lambda\cos(\varphi) & \lambda\operatorname{sen}(\varphi) & -\mu \end{vmatrix}$$

$$= \big(-\operatorname{sen}(\varphi),\ \cos(\varphi),\ 0\big) = \hat{\varphi}$$

$$\hat{\mu} \times \hat{\varphi} = \frac{1}{\sqrt{\lambda^2+\mu^2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \lambda\cos(\varphi) & \lambda\operatorname{sen}(\varphi) & -\mu \\ -\operatorname{sen}(\varphi) & \cos(\varphi) & 0 \end{vmatrix}$$

$$= \left(\frac{\mu}{\sqrt{\lambda^2+\mu^2}}\cos(\varphi),\ \frac{\mu}{\sqrt{\lambda^2+\mu^2}}\operatorname{sen}(\varphi),\ \lambda\right) = \hat{\lambda}$$

$$\hat{\varphi} \times \hat{\lambda} = \frac{1}{\sqrt{\lambda^2+\mu^2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\operatorname{sen}(\varphi) & \cos(\varphi) & 0 \\ \mu\cos(\varphi) & \mu\operatorname{sen}(\varphi) & \lambda \end{vmatrix}$$

$$= \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}}\cos(\varphi),\ \frac{\lambda}{\sqrt{\lambda^2+\mu^2}}\operatorname{sen}(\varphi),\ -\mu\right) = \hat{\mu}$$

Así, el sistema de coordenadas es ortogonal. (+0.5)

(c) Notemos que $\sqrt{x^2+y^2+z^2} = \sqrt{\lambda^2\mu^2 + \frac{1}{4}(\lambda^2-\mu^2)^2} = \frac{1}{2}(\lambda^2+\mu^2)$

y $z + \sqrt{x^2+y^2+z^2} = \lambda^2$. Así

$$\vec{F}(\lambda,\mu,\varphi) = \left(\frac{1}{\lambda^2}\cdot\frac{2\lambda\mu\cos(\varphi)}{\lambda^2+\mu^2},\ \frac{1}{\lambda^2}\cdot\frac{2\lambda\mu\operatorname{sen}(\varphi)}{\lambda^2+\mu^2},\ \frac{2}{\lambda^2+\mu^2}\right)$$

$$= \frac{2}{\lambda(\lambda^2+\mu^2)}\big(\mu\cos(\varphi),\ \mu\operatorname{sen}(\varphi),\ \lambda\big) \qquad (+0.5)$$

Con esto:

$$F_\lambda = \vec{F}\cdot\hat{\lambda} = \frac{2}{\lambda(\lambda^2+\mu^2)}\cdot\frac{\lambda^2+\mu^2}{\sqrt{\lambda^2+\mu^2}} = \frac{2}{\lambda\sqrt{\lambda^2+\mu^2}}$$

$$F_\mu = \vec{F}\cdot\hat{\mu} = 0$$

$$F_\varphi = \vec{F}\cdot\hat{\varphi} = 0 \qquad (+0.5)$$

(d) $\operatorname{div}(\vec{F}) = \frac{1}{h_\lambda h_\mu h_\varphi}\left(\frac{\partial}{\partial\lambda}(F_\lambda h_\mu h_\varphi) + \frac{\partial}{\partial\mu}(h_\lambda F_\mu h_\varphi) + \frac{\partial}{\partial\varphi}(h_\lambda h_\mu F_\varphi)\right)$

$$= \frac{1}{\lambda\mu(\lambda^2+\mu^2)} \cdot \frac{\partial}{\partial\lambda}\left(2\mu\right) = 0 \qquad (+1.0)$$

$$\operatorname{rot}(\vec{F}) = \frac{1}{h_\lambda h_\mu h_\varphi}\begin{vmatrix} h_\lambda \hat{\lambda} & h_\mu \hat{\mu} & h_\varphi \hat{\varphi} \\ \frac{\partial}{\partial\lambda} & \frac{\partial}{\partial\mu} & \frac{\partial}{\partial\varphi} \\ h_\lambda F_\lambda & h_\mu F_\mu & h_\varphi F_\varphi \end{vmatrix}$$

$$= \frac{1}{\lambda\mu(\lambda^2+\mu^2)}\begin{vmatrix} \sqrt{\lambda^2+\mu^2}\,\hat{\lambda} & \sqrt{\lambda^2+\mu^2}\,\hat{\mu} & \lambda\mu\,\hat{\varphi} \\ \frac{\partial}{\partial\lambda} & \frac{\partial}{\partial\mu} & \frac{\partial}{\partial\varphi} \\ \frac{2}{\lambda} & 0 & 0 \end{vmatrix} = 0 \qquad (+1.0)$$

P2 (a) Notamos que para $u \in [0, 2\pi]$, $v \in [0, \pi]$, las componentes de $\vec{\sigma}(u,v)$ verifican que

$$\frac{(a\cos(u)\operatorname{sen}(v))^2}{a^2} + \frac{(b\operatorname{sen}(u)\operatorname{sen}(v))^2}{b^2} + \frac{(c\cos(v))^2}{c^2}$$

$$= \cos^2(u)\operatorname{sen}^2(v) + \operatorname{sen}^2(u)\operatorname{sen}^2(v) + \cos^2(v) = 1. \quad (+0.5)$$

Vemos que S es la imagen de la superficie de la esfera unitaria a través del isomorfismo $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$(x,y,z) \mapsto (ax, bx, cx).$$

Notando que $\vec{\sigma}$ corresponde a la imagen a través de T de dicha esfera unitaria, se deduce que todo punto de S pertenece a la imagen de $\vec{\sigma}$. (+0.5)

Vemos finalmente que

$$\frac{\partial\vec{\sigma}}{\partial u} = (-a\operatorname{sen}(u)\operatorname{sen}(v), b\cos(u)\operatorname{sen}(v), 0)$$

$$\frac{\partial\vec{\sigma}}{\partial v} = (a\cos(u)\cos(v), b\operatorname{sen}(u)\cos(v), -c\operatorname{sen}(v))$$

Los cuales son l.i., con lo que la parametrización es regular. (+0.5)

(b) Notamos que si $(x_0, y_0, z_0) = \vec{\sigma}(u,v)$

$$\frac{\partial\vec{\sigma}}{\partial u} \times \frac{\partial\vec{\sigma}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\operatorname{sen}(u)\operatorname{sen}(v) & b\cos(u)\operatorname{sen}(v) & 0 \\ a\cos(u)\cos(v) & b\operatorname{sen}(u)\cos(v) & -c\operatorname{sen}(v) \end{vmatrix}$$

$$= (-bc\cos(u)\operatorname{sen}^2(v), -ac\operatorname{sen}(u)\operatorname{sen}^2(v), -ab\operatorname{sen}(v)\cos(v)),$$

vector que muestra la dirección normal interior a S. (+0.5)

Además

$$\vec{F}(x_0, y_0, z_0) = \vec{F}(\vec{\sigma}(u,v)) = \left(\frac{\cos(u)\operatorname{sen}(v)}{a}, \frac{\operatorname{sen}(u)\operatorname{sen}(v)}{b}, \frac{\cos(v)}{c}\right) \quad (+0.5)$$

$$= \frac{1}{abc\operatorname{sen}(v)}\left(bc\cos(u)\operatorname{sen}^2(v), ac\operatorname{sen}(u)\operatorname{sen}^2(v), ab\operatorname{sen}(v)\cos(v)\right)$$

De donde vemos que $\hat{n}$ y $\vec{F}$ son paralelos (+0.5)

(c) El plano tangente a S en $(x_0, y_0, z_0) = \vec{\sigma}(u,v)$ viene dado por

$$(\vec{x} - \vec{\sigma}(u,v)) \cdot \hat{n} = 0$$

Para hallar D, buscamos el valor de $t \geq 0$ tal que

$$(t\hat{n} - \vec{\sigma}(u,v)) \cdot \hat{n} = 0 \iff t = \vec{\sigma}(u,v) \cdot \hat{n} \qquad (+0.5)$$

Sabemos que

$$\hat{n} = -\frac{\frac{\partial \vec{\sigma}}{\partial u} \times \frac{\partial \vec{\sigma}}{\partial v}}{\left\| \frac{\partial \vec{\sigma}}{\partial u} \times \frac{\partial \vec{\sigma}}{\partial v} \right\|} = \frac{(bc\cos(u)\operatorname{sen}(v),\ ac\operatorname{sen}(u)\operatorname{sen}(v),\ ab\cos(v))}{\sqrt{b^2c^2\cos^2(u)\operatorname{sen}^2(v) + a^2c^2\operatorname{sen}^2(u)\operatorname{sen}^2(v) + a^2b^2\cos^2(v)}}$$

$$\vec{\sigma}(u,v) = (a\cos(u)\operatorname{sen}(v),\ b\operatorname{sen}(u)\operatorname{sen}(v),\ c\cos(v))$$

$$\Rightarrow \vec{\sigma}(u,v) \cdot \hat{n} = \frac{abc}{\sqrt{b^2c^2\cos^2(u)\operatorname{sen}^2(v) + a^2c^2\operatorname{sen}^2(u)\operatorname{sen}^2(v) + a^2b^2\cos^2(v)}}$$

Notando que $D = \|t\hat{n}\| = t$, lo anterior corresponde a D. $(+0.5)$

Por otra parte, vemos que

$$\vec{F}(\vec{\sigma}(u,v)) = \left(\frac{\cos(u)\operatorname{sen}(v)}{a},\ \frac{\operatorname{sen}(u)\operatorname{sen}(v)}{b},\ \frac{\cos(v)}{c} \right).$$

Así

$$\vec{F} \cdot \hat{n} = \frac{\frac{bc}{a}\cos^2(u)\operatorname{sen}^3(v) + \frac{ac}{b}\operatorname{sen}^2(u)\operatorname{sen}^3(v) + \frac{ab}{c}\operatorname{sen}(v)\cos(v)}{\left\| \frac{\partial \vec{\sigma}}{\partial u} \times \frac{\partial \vec{\sigma}}{\partial v} \right\|}$$

$$= \frac{\left(b^2c^2\cos^2(u)\operatorname{sen}^2(v) + a^2c^2\operatorname{sen}^2(u)\operatorname{sen}^2(v) + a^2b^2\cos^2(v)\right)\operatorname{sen}(v)}{abc\left\| \frac{\partial \vec{\sigma}}{\partial u} \times \frac{\partial \vec{\sigma}}{\partial v} \right\|}$$

$$= \frac{\sqrt{b^2c^2\cos^2(u)\operatorname{sen}^2(v) + a^2c^2\operatorname{sen}^2(u)\operatorname{sen}^2(v) + a^2b^2\cos^2(v)}}{abc}$$

$$\therefore D = \frac{1}{\vec{F} \cdot \hat{n}}. \qquad (+0.5)$$

(d) Tenemos que

$$\iint_S \frac{1}{D}\, dA = \int_0^{\pi}\int_0^{2\pi} \frac{\left\| \frac{\partial \vec{\sigma}}{\partial u} \times \frac{\partial \vec{\sigma}}{\partial v} \right\|}{abc\operatorname{sen}(v)} \cdot \left\| \frac{\partial \vec{\sigma}}{\partial u} \times \frac{\partial \vec{\sigma}}{\partial v} \right\| du\, dv$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{bc}{a}\cos^2(u)\,\mathrm{sen}^3(v) + \frac{ac}{b}\,\mathrm{sen}^2(u)\,\mathrm{sen}^3(v) + \frac{ab}{c}\,\mathrm{sen}(v)\cos^2(v)\,du\,dv$$

(+0.5)

Pero

$$\int_0^{\pi}\int_0^{2\pi} \cos^2(u)\,\mathrm{sen}^3(v) = \left(\int_0^{\pi} \mathrm{sen}(v) - \mathrm{sen}(v)\cos^2(v)\,dv\right)\left(\int_0^{2\pi} \frac{1+\cos(2u)}{2}\,du\right)$$

$$= \frac{4\pi}{3}$$

(+0.2)

$$\int_0^{\pi}\int_0^{2\pi} \mathrm{sen}^2(u)\,\mathrm{sen}^3(v)\,du\,dv = \left(\int_0^{\pi} \mathrm{sen}(v) - \mathrm{sen}(v)\cos^2(v)\,dv\right)\left(\int_0^{2\pi} \frac{1-\cos(2u)}{2}\,du\right)$$

$$= \frac{4\pi}{3}$$

(+0.2)

$$\int_0^{\pi}\int_0^{2\pi} \mathrm{sen}(v)\cos^2(v) = 2\pi\left(-\frac{1}{3}\cos^3(v)\right)\Big|_0^{\pi} = \frac{4\pi}{3}$$

(+0.2)

Con esto:

$$\iint_S \frac{1}{D}\,dA = \frac{4\pi}{3}\left(\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}\right)$$

(+0.4)