Aux 25/05

P1) Aux 7.

$f:\mathbb{R}^n \to \mathbb{R}$ $\quad f(x) = \underbrace{x^T A x}_{\in \mathbb{R}}$ $\quad A \in M_{n\times n}$, $x \in \mathbb{R}^n$

Pdq: f diff en $x_0 \in \mathbb{R}^n$ y encontramos $Df(x_0)$

Sol: Buscamos una función lineal $\ell(h)$ tq'

$$0 \leq \lim_{h\to 0} \frac{\| f(x_0+h) - f(x_0) - \ell(h) \|}{\|h\|_2} = 0$$

$$f(x_0+h) = (x_0+h)^T A (x_0+h)$$
$$= x_0^T A x_0 + x_0^T A h + \underline{h^T A x_0} + h^T A h$$

Sabemos que $h^T A x_0 \in \mathbb{R} \Rightarrow h^T A x_0 = (h^T A x_0)^T$
$= x_0^T A^T h$

$$\Rightarrow f(x_0+h) = \underbrace{x_0^T A x_0}_{f(x_0)} + \underbrace{x_0^T (A+A^T) h}_{\ell(h)} + \underbrace{h^T A h}_{\theta(h)} \leftarrow$$

$$\theta(h) = h^T A h = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \underline{h_i h_j}$$

(*) revisen la

Sabemos que $\forall i,j \in \{1,\dots,n\}$ $\quad |h_i||h_j| \leq |h_i|^2 + |h_j|^2$
$\leq \sum_{\ell=1}^{n} |h_\ell|^2 = \|h\|_2^2$

$$|\theta(h)| \leq \|h\|_2^2 \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| \qquad (*)$$

luego,

$$0 \leq \lim_{h\to 0} \frac{|f(x_0+h) - f(x_0) - \ell(h)|}{\|h\|_2} = \lim_{h\to 0} \frac{|\theta(h)|}{\|h\|_2}$$

$$\leq \lim_{h\to 0} \|h\| \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|}_{cte}$$

$$= 0$$

$$\ell(h) = \underbrace{x_0^T(A+A^T)}\, h$$

$$Df(\cdot) \neq \boxed{Df(x_0)(\cdot)}$$

$$\boxed{Df(x_0) = x_0^T(A+A^T)} \iff$$

$$\boxed{\ell(h) = Df(x_0)(h)} = Df(x_0)\, h$$

b) $g(x) = \|x\|_2^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$
$= x_1^2 + \cdots + x_n^2$

Notamos que g es igual a f de la parte a) con $A = I_n$

$$g(x) = x^T A x$$

$$Dg(x) = 2x^T = x^T(I_n + I_n^T)$$

$$= \begin{pmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{pmatrix} = \nabla g(x) = \begin{pmatrix} \frac{\partial g}{\partial x_1} \\ \vdots \\ \frac{\partial g}{x_n} \end{pmatrix}$$

$f: \mathbb{R}^2 \to \mathbb{R}$

P1) aux 8

$$f(x,y) = \begin{cases} \dfrac{x \sin(xy)}{x^2+y^2} & (x,y) \neq (0.0) \\ 0 & (x,y) = (0,0) \end{cases}$$

a) Para ver que es continua en el 0, sea $(x,y) \to (0,0)$

$|f(x,y) - 0| \to 0$

$$0 \leq |f(x,y)| = \frac{|x||\sin(xy)|}{|x^2+y^2|} \leq \frac{|x| \; |xy|}{x^2+y^2}$$

$|\sin(t)| \leq |t| \quad \forall t \in \mathbb{R}$

$$0 \leq |f(x,y)| \leq \frac{|x| \; |xy|}{x^2+y^2} \leq \frac{|x| \cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} = |x| \to 0$$

$|xy| \leq x^2+y^2$

$\Rightarrow f$ es continua en el $(0,0)$

b) $df((0,0);(e_1,e_2))$ con $(e_1,e_2) \neq (0,0)$

$$df((0,0);(e_1,e_2)) = \lim_{t\to 0} \frac{f(te_1,te_2) - \cancel{f(0.0)}^{0}}{t}$$

$$= \lim_{t \to 0} \frac{\cancel{t}e_1 \sin(t^2 e_1 e_2)}{\cancel{t}((te_1)^2 + (te_2)^2)}$$

$$= \lim_{t\to 0} \frac{e_1 \sin(t^2 e_1 e_2)}{t^2(e_1^2+e_2^2)} \cdot \frac{t^2 e_1 e_2}{t^2 e_1 e_2}$$

$$= \lim_{t\to 0} \frac{\cancel{t^2} e_1^2 e_2}{\cancel{t^2}(e_1^2+e_2^2)} \cdot \boxed{\frac{\sin(t^2 e_1 e_2)}{t^2 e_1 e_2}}$$

$\downarrow$ 1 (*)

$$\boxed{* \lim_{u\to 0} \frac{\sin(u)}{u} = 1}$$

$$= \frac{e_1^2 e_2}{e_1^2 + e_2^2} \quad //$$

c) $\frac{\partial f}{\partial x}(0,0) = df((0,0);(1,0)) = 0$

$\frac{\partial f}{\partial y}(0,0) = df((0,0);(0,1)) = 0$

luego, el candidato a diff. será $\underbrace{Df(0,0)}: \mathbb{R}^2 \to \mathbb{R}$

$$Df(0,0)[(x,y)] = L[(x,y)] = \underbrace{(0\ 0)}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

luego

$$0 \le \frac{|\overbrace{f((0,0)+(h_1,h_2))}^{f(h_1,h_2)} - \overbrace{f(0,0)}^{0} - \overbrace{L[(h_1,h_2)]}^{0}|}{\|(h_1,h_2)\|}$$

$$= \frac{|h_1|\,|\sin(h_1h_2)|}{(h_1^2+h_2^2)^{3/2}} \cdot \frac{|h_1h_2|}{|h_1h_2|} = \underbrace{\frac{|\sin(h_1h_2)|}{|h_1h_2|}}_{\downarrow\ 1} \cdot \underbrace{\frac{|h_1|^2|h_2|}{(h_1^2+h_2^2)^{3/2}}}_{*}$$

Tomamos $h_1 = h_2 = \frac{1}{n}$ sucesiones $n \in \mathbb{N}$

$$* = \frac{\frac{1}{n^3}}{\left(\frac{2}{n^2}\right)^{3/2}} = \frac{\frac{1}{\cancel{n^3}}}{\frac{(2)^{3/2}}{\cancel{n^3}}} = \frac{1}{2^{3/2}} \not\to 0$$

$\xrightarrow[n]{}$ $\Rightarrow f$ no es diff en $(0,0)$

P2

$f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} xy^2 \sin\left(\frac{1}{xy}\right) & x \cdot y \neq 0 \\ 0 & xy = 0 \end{cases}$$

a) caso $x \cdot y \neq 0$. luego

$$\frac{\partial f}{\partial x}(x,y) = y^2 \sin\left(\frac{1}{xy}\right) - \frac{y}{x} \cos\left(\frac{1}{xy}\right)$$

$$\frac{\partial f}{\partial y}(x,y) = 2xy \sin\left(\frac{1}{xy}\right) - \cos\left(\frac{1}{xy}\right)$$

·) $x = 0$ $y \neq 0$ derivada direccional

$$\frac{\partial f}{\partial x}(x,y) = \lim_{t \to 0} \frac{f(0+t, y) - \cancel{f(0,y)}^{0}}{t}$$

$$= \lim_{t \to 0} \frac{f(t,y)}{t} = \lim_{t \to 0} \frac{\cancel{t} y^2 \sin\left(\frac{1}{ty}\right)}{\cancel{t}}$$

$$= \lim_{t \to 0} y^2 \sin\left(\frac{1}{ty}\right)$$

comprobar $= \nexists$

··) $x \neq 0$ $y = 0$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{t \to 0} \frac{\cancel{f(x+t, 0)}^{0} - \cancel{f(x,0)}^{0}}{t}$$

$$= \lim_{t \to 0} \frac{0}{t} = 0 \quad //$$

iii) $x=0 \wedge y=0$

$$\frac{\partial f}{\partial x}(x,y) = \frac{f(t,0) - f(0,0)}{t}$$

$$= 0$$

ahora vamos con $\frac{\partial f}{\partial y}$

i) $x \neq 0 \wedge y=0$ $\qquad \frac{\partial f}{\partial y}(x,y) = 0$

ii) $x=0 \wedge y \neq 0$ $\qquad \frac{\partial f}{\partial y}(x,y) = 0$

iii) $x=0 \wedge y=0$ $\qquad \frac{\partial f}{\partial y}(x,y) = 0$

b) sabemos que $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$

Entonces el candidato

$Df(0,0) : \mathbb{R}^2 \to \mathbb{R}$

$$L[(x,y)] = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\lim_{\|(h_1,h_2)\|\to 0} \frac{\left| f((0,0)+(h_1,h_2)) - \cancel{f(0,0)}^{=0} - \cancel{L(h_1,h_2)}^{=0} \right|}{\|(h_1,h_2)\|}$$

$$= \lim_{\|(h_1,h_2)\|\to 0} \left| \frac{h_1 h_2^2 \sin\left(\frac{1}{h_1 h_2}\right)}{\sqrt{h_1^2+h_2^2}} \right|$$

$\leq$ $2|ab| \leq a^2+b^2$

$$\leq \lim_{\|(h_1,h_2)\|\to 0} \left| \frac{h_2\sqrt{h_1^2+h_2^2}}{2} \underbrace{\sin\left(\frac{1}{h_1 h_2}\right)}_{\leq 1} \right|$$

$$= \lim_{\|(h_1,h_2)\|\to 0} \left| h_2 \sqrt{h_1^2+h_2^2} \right| = 0$$

$\Rightarrow$ f es diff en $(0,0)$ ■

c) ¿Es $\frac{\partial f}{\partial y}$ continua en (0.0)?

(Dbf!) $$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x,y) = \lim_{(x,y)\to(0,0)} \underbrace{2xy\sin\left(\frac{1}{xy}\right) - \cos\left(\frac{1}{xy}\right)}$$

$= 0$

Si pensamos en el caso

$x = y$

$$\frac{\partial f}{\partial y}(x,x) = 2x^2 \sin\left(\frac{1}{x^2}\right) - \cos\left(\frac{1}{x^2}\right)$$

$$\neq 0 \implies \frac{\partial f}{\partial y} \text{ no es continua}$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

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