$$\sum_{k=1}^{n} k\left(\ln\left(\underbrace{1+\tfrac{1}{k}}_{\frac{k+1}{k}}\right)\right) = \sum_{k=1}^{n} k\cdot\ln\left(\frac{k+1}{k}\right)$$

el problema es calcular esa suma

$$= \sum_{k=1}^{n} k\cdot\left(\ln(k+1)-\ln(k)\right) = \sum_{k=1}^{n} \underline{k\cdot\ln(k+1)} - k\cdot\ln(k) + \overbrace{\left(\underline{\ln(k+1)}-\ln(k+1)\right)}^{\text{sumar cero}}$$

$$= \sum_{k=1}^{n} \underbrace{(k+1)\cdot\ln(k+1)}_{\ln(k+1)+k\cdot\ln(k+1)} - k\cdot\ln(k) - \ln(k+1)$$

$$= \underbrace{\sum_{k=1}^{n} (k+1)\cdot\ln(k+1) - k\cdot\ln(k)} - \sum_{k=1}^{n}\ln(k+1)$$

$$\overset{S.T.}{=} (n+1)\cdot\ln(n+1) - \cancel{1\cdot\ln(1)}^{\,0} - \sum_{k=1}^{n}\ln(k+1)$$

$= ?$

$\ln((n+1)!)$

$$\ln\left(\frac{(k+1)!}{k!}\right) = \ln((k+1)!) - \ln(k!)$$

$$\overset{S.T}{=} \ln((n+1)!) - \underbrace{\ln(1!)}_{0}$$

$$= \ln((n+1)!)$$

$$(k+1)! = k!\cdot(k+1)$$

$$\frac{(k+1)!}{k!} = (k+1)$$

¿qué pasa si $n=2$?

$$\log(a\cdot b) = \log(a) + \log(b)$$

$\downarrow$

$$\log(a\cdot b\cdot c) = \log(a) + \log(b) + \log(c)$$

$\downarrow$

$$\ln((n+1)\cdot n\cdot\ldots\cdot 2) = \sum_{k=1}^{n}\ln(k+1) = \ln(n+1) + \ln(n) + \ldots + \ln(2)$$

$\parallel$

$$\ln((n+1)\cdot n\cdot\ldots\cdot 2\cdot 1) = \ln((n+1)!)$$

$\therefore$ La suma final queda: $(n+1)\cdot\ln(n+1) - \ln((n+1)!)$

P2) Sea $k_0 \in \mathbb{N}$, $n \in \mathbb{N}$ impar. Hay que demostrar que la suma de los primeros n naturales desde k_0 es divisible por n. $(k_0, k_0+1, \dots)$

$$\rightarrow k_0 + (k_0+1) + (k_0+2) + (k_0+3) + \dots + (k_0 + \quad)$$

$$i) = \sum_{i=k_0}^{n} i = k_0 + (k_0+1) + (k_0+2) + \dots + \underline{\underline{n}} \quad \times$$

$$ii) = \sum_{i=0}^{n} (k_0 + i) = (k_0) + (k_0+1) + (k_0+2) + \dots + (k_0+n) \quad \times$$

$$iii) = \sum_{i=k_0}^{n-1} i \quad \times$$

$$iv) = \sum_{i=0}^{n-1} (k_0+i) = k_0 + (k_0+1) + \dots + (k_0+n-1)$$

$$= \sum_{i=k_0}^{k_0+n-1} i$$

también serviría. (pero no nos gusta tanto)

Seguimos trabajando la suma:

$$\sum_{i=0}^{n-1} (k_0+i) = \sum_{i=0}^{n-1} k_0 + \sum_{i=0}^{n-1} i$$

$$= k_0 \cdot \underbrace{\left(\sum_{i=0}^{n-1} 1\right)} + \underbrace{\sum_{i=0}^{n-1} i}$$

$$= k_0 \cdot [n - \cancel{1} - 0 + \cancel{1}] + \left[\frac{(n-1)\cdot(\cancel{n} - \cancel{1} + \cancel{1})}{2}\right]$$

$$= \underbrace{k_0 \cdot n}_{\text{div} \times n} + \underbrace{\frac{n \cdot (n-1)}{2}}_{?}$$

n es impar

$\Rightarrow n-1$ es par

$\Rightarrow \frac{n-1}{2}$ es $\mathbb{N}$

Luego $n \cdot \frac{(n-1)}{2}$ es div $\times n$

Finalmente, la suma es divisible por n y concluimos lo pedido.

P3) Justifique la segunda igualdad

$$S = \sum_{k=0}^{n} \binom{2n+1}{k} = \sum_{k=0}^{n} \binom{2n+1}{n+k+1}$$

Propiedad:

$$\binom{n}{m} = \binom{n}{n-m} \rightarrow \frac{n!}{(n-m)! \cdot m!} = \frac{n!}{(n-(n-m))! \cdot (n-m)!}$$

$$= \binom{n}{n-m}$$

$$\binom{a}{b} = \binom{a}{a-b}$$

$a = 2n+1$

$b = k$

$$\rightarrow \binom{2n+1}{k} = \binom{2n+1}{2n+1-k}$$

$$k=0 \rightarrow \binom{2n+1}{0} \rightarrow \binom{2n+1}{2n+1} = \binom{2n+1}{n+k+1} \text{ con } k=n$$

$$k=1 \rightarrow \binom{2n+1}{1} \rightarrow \binom{2n+1}{2n} = \binom{2n+1}{n+k+1} \text{ con } k=n-1.$$

Definimos $k' = n-k \rightarrow \binom{2n+1}{2n+1-k} = \binom{2n+1}{n+n-k+1}$

$$= \binom{2n+1}{n+k'+1}$$

$$\sum_{k=0}^{n}\binom{2n+1}{k} = \sum_{k=0}^{n}\binom{2n+1}{2n+1-k}$$

$$= \sum_{k=0}^{n}\binom{2n+1}{n+k'+1}$$

$k=0 \rightarrow k' = ? = n$

$k=1 \rightarrow k' = n-1$

$\vdots$

$k=n \rightarrow k'=0$

$$= \sum_{k'=0}^{n}\binom{2n+1}{n+k'+1}$$

k' es solo un nombre, lo reemplazamos por k.

$$= \sum_{k=0}^{n}\binom{2n+1}{n+k+1}$$

dem $\not{p}$: $S = 2^{2n}$

hint: Calcular $\sum_{k=0}^{n}\binom{2n+1}{k} + \sum_{k=0}^{n}\binom{2n+1}{n+k+1}$

$$\underbrace{\sum_{k=0}^{n}\binom{2n+1}{k} + \sum_{k=0}^{n}\binom{2n+1}{n+k+1}}_{} = 2\cdot S$$

nos gustaría que fuera $2^{2n+1} \Rightarrow S = 2^{2n}$

$$= 2\cdot \sum_{k=0}^{n}\binom{2n+1}{k} 1^{k}\cdot 1^{2n+1-k}$$

? X

no apaña mucho

Otra idea! <u>Cambio de índice:</u>

$$\underbrace{\sum_{k=0}^{n}\binom{2n+1}{k}} + \sum_{k=0}^{n}\binom{2n+1}{n+k+1}$$

sumamos n+1 afuera para restar n+1 adentro

$$\sum_{k=n+1}^{2n+1}\binom{2n+1}{k-\cancel{(n+1)}\cancel{+n+1}}$$

$$= \sum_{k=n+1}^{2n+1}\binom{2n+1}{k}$$

$$= \sum_{k=0}^{n}\binom{2n+1}{k} + \sum_{k=n+1}^{2n+1}\binom{2n+1}{k} = \sum_{k=0}^{2n+1}\binom{2n+1}{k}$$

$$= \sum_{k=0}^{2n+1}\binom{2n+1}{k}\cdot 1^{k}\cdot 1^{2n+1-k}$$

Por Teo. Binomio $=$ $(1+1)^{2n+1} = 2^{2n+1}$

$$\Rightarrow \cancel{2}\cdot S = \cancel{2^{2n+1}} \Rightarrow S = 2^{2n}$$