MA1101 - Introducción al Álgebra
Auxiliar 8

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Resumen

Sumatoria

Es solo notación para comprimir sumas muy largas:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

Sumas conocidas

- **Suma de unos:**

$1 + 1 + 1 + 1 + \ldots + 1$?

$k=m$ $\quad$ $k=n$

$$\sum_{k=m}^{n} 1 = n - m + 1$$

$n - m + 1$

- **Sacar el escalar:**

$$\sum_{k=m}^{n} \lambda \cdot a_k = \lambda \cdot \sum_{k=m}^{n} a_k$$

Resumen

Sumas conocidas

- **Separar la suma:**

$$\sum_{k=m}^{n} (a_k \pm b_k) = \sum_{k=m}^{n} a_k \pm \sum_{k=m}^{n} b_k$$

- **Traslación de índices:**

$$\sum_{k=m}^{n} a_k = \sum_{k=m-s}^{n-s} a_{k+s} = \sum_{k=m+s}^{n+s} a_{k-s}$$

- **Separar los índices:** $(a_m + a_{m+1} + \dots + a_s) + (a_{s+1} + \dots + a_n)$

$$\sum_{k=m}^{n} a_k = \sum_{k=m}^{s} a_k + \sum_{k=s+1}^{n} a_k \quad \text{para } m \leq s < n$$

Resumen

Sumas conocidas

- **Suma Telescópica:**

$$\sum_{i=m}^{n} \frac{1}{a_k} - \frac{1}{a_{k+1}} = \frac{1}{a_m} - \frac{1}{a_{n+1}}$$

$$\sum_{k=m}^{n} (a_k - a_{k+1}) = (a_m - a_{n+1}).$$

- **Suma de los primeros n naturales:**

$$\sum_{k=1}^{n} k \overset{\text{lo mismo}}{=} \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

- **Suma de los primeros n cuadrados**

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Resumen

$$\sum_{i=1}^{n} \frac{1}{k^2} ?$$

Sumas conocidas

- **Suma de los primeros n cubos**

$$\sum_{k=0}^{n} k^3 = (\frac{n(n+1)}{2})^2$$

Obs: para las 3 sumas anteriores, si parten de 0 o de 1 es la misma fórmula (Piensen que si suman el cero en realidad no alteran nada)

- **Geometrica:**

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$$

para $r \neq 1$

Sumatorias

P1.

Probar por inducción que $(n \geq 1)$

$$\sum_{k=n+1}^{2n} \frac{1}{k} = \sum_{j=1}^{2n} \frac{(-1)^{j+1}}{j}$$

C.B. $n=1$: $\displaystyle\sum_{k=1+1}^{2\cdot 1} \frac{1}{k} \overset{\text{me gustaría}}{=} \sum_{j=1}^{2\cdot 1} \frac{(-1)^{j+1}}{j}$

L.I. $\displaystyle\sum_{k=2}^{2} \frac{1}{k} = 1/2$

L.D. $\displaystyle\sum_{j=1}^{2} \frac{(-1)^{j+1}}{j} = \frac{(-1)^{1+1}}{1} + \frac{(-1)^{2+1}}{2} = 1 + (-1/2) = 1/2$

Problemas

H.I. $$\sum_{k=n+1}^{2n} 1/k = \sum_{j=1}^{2n} \frac{(-1)^{j+1}}{j}$$

Demostración

P.I. $$\sum_{k=(n+1)+1}^{2(n+1)} 1/k \overset{?}{=} \sum_{j=1}^{2(n+1)} \frac{(-1)^{j+1}}{j}$$

$$\sum_{k=n+2}^{2n+2} 1/k = \left(1/_{n+2} + 1/_{n+3} + \dots + 1/_{2n} \right) + \left(1/_{2n+1} + 1/_{2n+2} \right)$$

$$= \sum_{k=n+2}^{2n} 1/k + \sum_{k=2n+1}^{2n+2} 1/k + \left(1/_{n+1} - 1/_{n+1} \right) \qquad \sum_{k=n+1}^{n+1} 1/k$$

$$= \sum_{k=n+1}^{2n} 1/k + \sum_{k=2n+1}^{2n+2} 1/k - 1/_{n+1}$$

Problemas

Demostración $\displaystyle\sum_{k=n+1}^{2n} 1/k + \sum_{k=2n+1}^{2n+2} 1/k - 1/n+1$, por H.I.

$$= \sum_{j=1}^{2n} \frac{(-1)^{j+1}}{j} + \underbrace{\sum_{k=2n+1}^{2n+2} 1/k - 1/n+1}_{\text{MOTIVACIÓN: son los términos cuando } j=2n+1 \text{ y } 2n+2.} \overset{?}{=} \sum_{j=1}^{2(n+1)} \frac{(-1)^{j+1}}{j}$$

$$j = 2n+1, \quad \frac{(-1)^{j+1}}{j} ? \longrightarrow \frac{(-1)^{\overset{\text{par}}{(2n+1)+1}}}{2n+1} = \frac{1}{2n+1}$$

$$j = 2n+2, \quad \frac{(-1)^{j+1}}{j} ? \longrightarrow \frac{(-1)^{2n+2+1}}{2n+2} = \frac{-1}{2n+2}$$

Problemas

Demostración $\displaystyle\sum_{k=2n+1}^{2n+2} 1/k - 1/(n+1)$

$$= \frac{1}{2n+1} + \underbrace{\frac{1}{2n+2} - \frac{1}{n+1}}$$

$$\frac{2}{2(n+1)} = \frac{2}{2n+2}$$

$$= \frac{1-2}{2n+2} = \frac{-1}{2n+2}$$

$$\Rightarrow \sum_{j=1}^{2n} \frac{(-1)^{j+1}}{j} + \sum_{k=2n+1}^{2n+2} 1/k - 1/(n+1) = \sum_{j=1}^{2(n+1)} \frac{(-1)^{j+1}}{j}$$

Concluyendo, por inducción.

Sumatorias

P2.

Calcule la siguiente suma

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k(k+1)}(\sqrt{k+1}+\sqrt{k})} \Bigg/ \cdot \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}}$$

$$= \sum_{k=1}^{n} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}\cdot(\cancel{k}+1-\cancel{k})}$$

$$= \sum_{k=1}^{n} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}} = \sum_{k=1}^{n} \frac{\sqrt{k+1}}{\sqrt{k(k+1)}} - \frac{\sqrt{k}}{\sqrt{k(k+1)}}$$

Problemas

Demostración

$$\sum_{k=1}^{n}\left(\frac{\cancel{\sqrt{k+1}}}{\cancel{\sqrt{k(k+1)}}} - \frac{\cancel{\sqrt{k}}}{\cancel{\sqrt{k(k+1)}}}\right) = \sum\left(\frac{\cancel{\sqrt{k+1}}}{\sqrt{k}\,\cancel{\sqrt{k+1}}} - \frac{\cancel{\sqrt{k}}}{\cancel{\sqrt{k}}\,\sqrt{k+1}}\right)$$

$$= \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) \overset{\text{Suma telesc.}}{=} \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{n+1}}$$

$$\sum_{k=a}^{b} C_k - C_{k+1} = C_a - C_{b+1}$$

Sumatorias

P3.

Sabiendo que $(\forall n)$

$$\sum_{i=1}^{n} u_i = 2n^2 + 3n$$

- fact.
- ~~separar suma~~
- ~~cambio ind.~~
- separ. índices

Calcule el valor de las siguientes expresiones

- $\sum_{i=n+1}^{2n} u_i \longrightarrow \sum_{i=1}^{2n} u_i = 2\cdot(2n)^2 + 3\cdot(2n)$
- u_n

$$\sum_{i=1}^{2n} u_i = \sum_{i=1}^{n} u_i + \sum_{i=n+1}^{2n} u_i$$

Problemas

Demostración

$$\sum_{i=n+1}^{2n} u_i = \underbrace{\sum_{i=1}^{2n} u_i}_{8n^2+6n} - \underbrace{\sum_{i=1}^{n} u_i}_{2n^2+3n}$$

$$\Rightarrow \sum_{i=n+1}^{2n} u_i = 8n^2+6n-(2n^2+3n)$$

$$= 6n^2+3n$$

$$u_n = \underbrace{\sum_{i=1}^{n} u_i}_{2n^2+3n} - \underbrace{\sum_{i=1}^{n-1} u_i}_{2(n-1)^2+3(n-1)} = 2n^2+3n-(2n^2-4n+2+3n-3)$$

$$= 4n+1$$

Problemas

Demostración

$$\sum_{i=a}^{b \to b+1\ ???} c_i$$

sumar 0.

$$\to \sum_{i=a}^{b} c_i + (c_{b+1} - c_{b+1})$$

$$= \left(\sum_{i=a}^{b+1} c_i\right) - c_{b+1}$$

Sumatorias

P4. $n \geq 1,\ r \neq 1$:

$$S_n = \sum_{k=1}^{n} k r^k$$

Demuestre que: (sin inducción)

i) $$S_n = r\left(S_n - n r^n\right) + \sum_{k=0}^{n-1} r^{k+1}$$

Problemas

Demostración

Sumatorias

Tips.

$k! = k(k-1)!$, $2^k = 2^{k-1} + 2^{k-1}$

P5.

Calcule la siguiente suma

$$\sum_{k=1}^{n} \frac{k^2+5k+5}{(k+4)!} = \frac{(k+2)}{(k+3)!} - \frac{(k+3)}{(k+4)!}$$

$$\sum_{k=1}^{n} \frac{k+3}{(k+3)!} - \frac{k+4}{(k+4)!} = \sum_{k=1}^{n} \frac{(k+4)(k+3) - (k+4)}{(k+4)!}$$

$$= \sum_{k=1}^{n} \frac{k^2+7k+12-k-4}{(k+4)!} = \sum_{k=1}^{n} \frac{k^2+6k+8}{(k+4)!}$$

Problemas

Demostración

$$\sum_{k=1}^{n} \frac{k^2+6k+8}{(k+4)!} = \sum_{k=1}^{n} \frac{k^2+5k+5}{(k+4)!}$$

$$+ \sum_{k=1}^{n} \frac{k+3}{(k+4)!}$$

$$\sum_{k=1}^{n} \frac{k+3}{(k+4)!} = \sum_{k=1}^{n} \frac{k+3+1-1}{(k+4)!} = \sum_{k=1}^{n} \cancel{\frac{k+4}{(k+4)!}} - \frac{1}{(k+4)!}$$

$$= \sum_{k=1}^{n} \frac{1}{(k+3)!} - \frac{1}{(k+4)!} \overset{S.T.}{=} \frac{1}{(1+3)!} - \frac{1}{(n+4)!}$$

Problemas

Demostración

$$\sum_{k=1}^{n} \frac{k^2+6k+8}{(k+4)!} = \sum_{k=1}^{n} \frac{k^2+5k+5}{(k+4)!} + \sum_{k=1}^{n} \frac{k+3}{(k+4)!}$$

$$-$$

$$\sum_{k=1}^{n} \frac{k^2+6k+8}{(k+4)!} - \left(\frac{1}{4!} - \frac{1}{(n+4)!}\right) = \sum_{k=1}^{n} \frac{k^2+5k+5}{(k+4)!}$$

$$= \left(\sum_{k=1}^{n} \frac{k+3}{(k+3)!} - \frac{k+4}{(k+4)!}\right) - \left(\frac{1}{24} - \frac{1}{(n+4)!}\right)$$

S.T.

$$= \frac{\cancel{4}}{\cancel{4!}} - \frac{\cancel{n+4}}{\cancel{(n+4)!}} - \left(\frac{1}{24} - \frac{1}{(n+4)!}\right) = \frac{1}{3!} - \frac{1}{(n+3)!} - \frac{1}{24} + \frac{1}{(n+4)!}$$

Sumatorias

P6.

Calcule la suma de los primeros n términos de la siguiente sucesión:

$$\frac{1\cdot 2^1}{3!}, \frac{2\cdot 2^2}{4!}, \frac{3\cdot 2^3}{5!}, \frac{4\cdot 2^4}{6!}, \ldots$$

¿Qué hago? . . . ¡¡? ????

$$\sum_{k=1}^{n} \frac{k\cdot 2^k}{(k+2)!} = \sum_{k=1}^{n} \frac{[(k+2)-2]2^k}{(k+2)!} =$$

$$= \sum_{k=1}^{n} \frac{\cancel{(k+2)}2^k}{\cancel{(k+2)}!} - \frac{2\cdot 2^k}{(k+2)!} = \sum_{k=1}^{n} \frac{2^k}{(k+1)!} - \frac{2^{k+1}}{(k+2)!} = \frac{2^1}{(1+1)!} - \frac{2^{n+1}}{(n+2)!}$$

Problemas

Demostración

$$\sum_{k=1}^{n} \frac{2^k}{(k+1)!} - \frac{2^{k+1}}{(k+2)!} \qquad a_k - a_{k+1}$$

$k=1$: $\dfrac{2^1}{(1+1)!} - \cancel{\dfrac{2^{1+1}}{(1+2)!}}$

$k=2$: $\cancel{\dfrac{2^2}{(2+1)!}} - \cancel{\dfrac{2^3}{4!}}$

$k=3$ $\cancel{\dfrac{2^3}{4!}} - \dfrac{2^4}{5!}$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j}$$

$$\sum_{j=1}^{m} \sum_{i=1}^{n} a_{i,j}$$