Pauta Aux 18:

Ejemplos Euler-Lagrange:

1) **Clásico 1D en cartesianas**: $K = \frac{1}{2} m \dot{x}^2$ y $U = U(x)$

$$\Rightarrow \boxed{\mathcal{L} = K - U} = \frac{1}{2} m \dot{x}^2 - U(x)$$

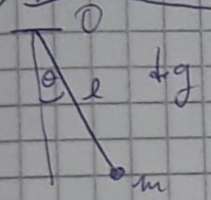
$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \quad y \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x}$$

$$\left.\right\} \boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0}$$

$$\iff \boxed{m\ddot{x} + \frac{\partial U}{\partial x} = 0} \quad //$$

2) **Péndulo simple**



$$|\vec{v}| = l\dot{\theta} \Rightarrow K = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U(\theta) = mgy = -mgl\cos\theta$$

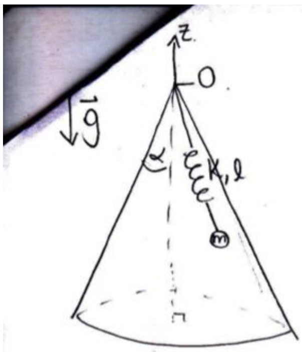
$$\therefore \boxed{\mathcal{L} = K - U} = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgl\,\text{sen}\,\theta$$

$$\left.\right\} \boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0}$$

$$\therefore \cancel{m}l^{\cancel{2}}\ddot{\theta} + \cancel{m}g\cancel{l}\,\text{sen}\,\theta = 0 \iff \boxed{\ddot{\theta} + \frac{g}{l}\,\text{sen}\,\theta = 0}$$



Aux 18

Utilizando cilíndricas:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

$$K = \frac{1}{2}m\left(\dot{r}^2 + \dot{\phi}^2 r^2 + \dot{z}^2\right)$$

$$U = mgz + \frac{1}{2}k(L-\ell)^2$$

Ahora:

$$r = L\,\mathrm{sen}(\alpha) \qquad \tan(\alpha) = \frac{r}{h} \Rightarrow z = -\frac{r}{\tan(\alpha)}$$

$$L = \frac{r}{\mathrm{sen}(\alpha)}$$

Entonces:

$$U = -mg\frac{r}{\tan(\alpha)} + \frac{1}{2}k\left(\frac{r}{\mathrm{sen}(\alpha)} - \ell\right)^2$$

b) Ecuación Euler-Lagrange:

$$(1)\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \qquad (2)\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

Luego el lagrangiano

$$\mathcal{L} = K - U = \frac{1}{2}m\left(\dot{r}^2 + \dot{\phi}^2 r^2 + \frac{\dot{r}^2}{\tan^2(\alpha)}\right) + \frac{mgr}{\tan(\alpha)} - \frac{k}{2}\left(\frac{r}{\mathrm{sen}(\alpha)} - \ell\right)^2$$

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2\left(1 + \frac{1}{\tan^2(\alpha)}\right) + \dot{\phi}^2 r^2\right) + \frac{mgr}{\tan(\alpha)} - \frac{k}{2}\left(\frac{r}{\mathrm{sen}(\alpha)} - \ell\right)^2$$

$$\cdot\ \frac{\partial \mathcal{L}}{\partial r} = \frac{mg}{\tan(\alpha)} - K\left(\frac{r}{\text{sen}(\alpha)} - l\right)\frac{1}{\text{sen}(\alpha)} + 2r\dot{\phi}^2.$$

$$\cdot\ \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}\left(1 + \frac{1}{\tan^2(\alpha)}\right) \rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = m\ddot{r}\left(1 + \frac{1}{\tan^2(\alpha)}\right)$$

$$\cdot\ \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\cdot\ \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi} \rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi}$$

Entonces:

$$(1):\ m\ddot{r}\left(1 + \frac{1}{\tan^2(\alpha)}\right) - \frac{mg}{\tan(\alpha)} + \frac{K}{\text{sen}(\alpha)}\left(\frac{r}{\text{sen}(\alpha)} - l\right) - 2r\dot{\phi}^2 = 0$$

$$(2):\ mr^2\ddot{\phi} + 2r\dot{r}m\dot{\phi} = 0$$

Esto significa: $\frac{d}{dt}\left(mr^2\dot{\phi}^2\right) = 0 \Rightarrow mr^2\dot{\phi}^2 = c$ cte.

$$\dot{\phi}^2 = \frac{c}{mr^2} \quad (3).$$

Utilizando (3) en (1):

$$m\ddot{r}\left(1 + \frac{1}{\text{Tan}^2(\alpha)}\right) - \frac{mg}{\tan(\alpha)} + \frac{K}{\text{sen}(\alpha)}\left(\frac{r}{\text{sen}(\alpha)} - l\right) - \frac{2c}{mr^3} = 0. \quad (4)$$

Solución P2: La energía cinética del sistema es: $K = \frac{m}{2}(\dot{y}_1^2 + \dot{y}_2^2)$. Si el sistema está en una configuración y_1 e y_2, el largo de la primera cuerda es $2L - y_2$ y el largo de la segunda cuerda es $y_2 + y_2 - y_1$. Entonces la primera cuerda se estira $2L - y_2 - L$ y la segunda cuerda se estira $2y_2 - y_1 - 3L/2$. Luego, la energía potencial almacenada en las cuerdas es:

$$U_{\text{cuerdas}}(y_1, y_2) = \frac{k}{2}(y_2 - L)^2 + \frac{k}{2}(2y_2 - y_1 - 3L/2)^2. \tag{3}$$

Incluyendo la energía potencial gravitacional, se encuentra que el Lagrangiano del sistema es

$$L = \frac{m}{2}(\dot{y}_1^2 + \dot{y}_2^2) - \frac{k}{2}(y_2 - L)^2 - \frac{k}{2}(2y_2 - y_1 - 3L/2)^2 - mgy_1 - mgy_2. \tag{4}$$

Las ecuaciones de movimiento son:

$$\ddot{y}_1 + \frac{k}{m}(y_1 - 2y_2 + 3L/2) + g = 0, \qquad \ddot{y}_2 + \frac{k}{m}(y_2 - L) + 2\frac{k}{m}(2y_2 - y_1 - 3L/2) + g = 0. \tag{5}$$

Las posiciones de equilibrio se determinan mediante $\ddot{y}_1 = \ddot{y}_2 = 0$. Se encuentra: $\bar{y}_1 = \frac{1}{2}L - 7g/\omega_0^2$, $\bar{y}_2 = L - 3g/\omega_0^2$, donde $\omega_0 = \sqrt{k/m}$. Luego:

$$\ddot{x}_1 + \omega_0^2(x_1 - 2x_2) = 0, \qquad \ddot{x}_2 + \omega_0^2(5x_2 - 2x_1) = 0. \tag{6}$$

La matriz de frecuencias tiene las siguientes componentes $\Omega_{11} = \omega_0^2$, $\Omega_{12} = \Omega_{21} = -2\omega_0^2$ y $\Omega_{22} = 5\omega_0^2$. Las frecuencias propias son: $\omega_I^2 = (3 + 2\sqrt{2})\omega_0^2$ y $\omega_{II}^2 = (3 - 2\sqrt{2})\omega_0^2$. Los vectores propios asociados son (no normalizados):

$$\vec{e}_I = (1 - \sqrt{2}, 1) \simeq (-0.4, 1), \qquad \vec{e}_{II} = (1 + \sqrt{2}, 1) \simeq (2.4, 1). \tag{7}$$