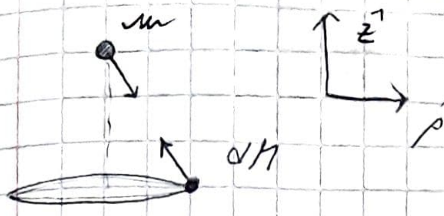

a) $d\vec{F}_{m\,dM} = -G\,m\,dM \cdot \dfrac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$

$\vec{r} = z\hat{z}$
$\vec{r}\,' = R\hat{\rho}$ $\Rightarrow d\vec{F}_{m\,dM} = -G\,m\,dM \cdot \dfrac{(z\hat{z}-R\hat{\rho})}{[z^2+R^2]^{3/2}}$

Luego $\vec{F}_{mM} = \displaystyle\int_{\substack{\text{todo el}\\ \text{anillo}}} -\frac{Gm\,(z\hat{z}-R\hat{\rho})}{(z^2+R^2)^{3/2}}\,dM$

$\rho_{lineal} = \lambda = \dfrac{dM}{dl} = \dfrac{M}{2\pi R} \Rightarrow dM = \lambda dl = \dfrac{M}{2\pi R}\,dl$

$dl = R\,d\theta \Rightarrow dM = \dfrac{M}{2\pi}\,d\theta.$

Luego: $\vec{F}_{mM} = \displaystyle\int_0^{2\pi} -\frac{Gm\,(z\hat{z}-R\hat{\rho})}{(z^2+R^2)^{3/2}}\cdot\frac{M}{2\pi}\,d\theta$

$= -\dfrac{GmM}{2\pi}\displaystyle\int_0^{2\pi}\frac{(z\hat{z}-R\hat{\rho})}{(z^2+R^2)^{3/2}}\,d\theta.$

$$\vec{F}_{mM} = -\frac{GmM}{2\pi}\left[\int_0^{2\pi}\frac{z\hat{z}}{(z^2+R^2)^{3/2}}\,d\theta - \underbrace{\int_0^{2\pi}\frac{R\hat{\rho}}{(z^2+R^2)^{3/2}}\,d\theta}_{=0}\right]$$

Ya que $\hat{\rho} = \sin\theta\,\hat{x} + \cos\theta\,\hat{y}$ integrado de $0;2\pi$ es nulo.

$\therefore\ \vec{F}_{mM} = -\dfrac{GmM}{2\pi}\displaystyle\int_0^{2\pi}\underbrace{\frac{z\hat{z}}{(z^2+R^2)^{3/2}}}_{\text{¡No depende de }\theta!}\,d\theta$

$= -\dfrac{GmM}{\cancel{2\pi}}\cdot\dfrac{z\hat{z}}{(z^2+R^2)^{3/2}}\cdot\cancel{2\pi}$

$= -\dfrac{GmM\,z\hat{z}}{(z^2+R^2)^{3/2}}$ ✓.

b) $W = -\displaystyle\int_{\vec{r}_0}^{\vec{r}_f}\vec{F}\cdot d\vec{r}$; $d\vec{r} = dz\,\hat{z}$
$\vec{r}_0 = R\hat{z}$
$\vec{r}_f = 0$

$= -\displaystyle\int_R^0 -\frac{GmM}{(z^2+R^2)^{3/2}}\,z\hat{z}\cdot dz\,\hat{z}$

$= GmM\displaystyle\int_R^0\frac{z\,dz}{(z^2+R^2)^{3/2}} = GmM\int_0^R\frac{z\,dz}{R^3(z^2/R^2+1)^{3/2}}$

Cambio de variable: $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \tan^2\theta + 1 = \dfrac{1}{\cos^2\theta}$

$$\therefore \frac{z}{R} = \tan\theta \Rightarrow \frac{dz}{R} = \frac{d\theta}{\cos^2\theta}$$

$$\therefore W = \frac{GmM}{R^3} \int_{\pi/4}^{0} \frac{R \cdot \tan\theta \cdot \frac{R\,d\theta}{\cos^2\theta}}{(\tan^2\theta + 1)^{3/2}}$$

$$= \frac{GmM}{R} \int_{\pi/4}^{0} \frac{\frac{\operatorname{sen}\theta}{\cos^3\theta}}{\frac{1}{\cos^3\theta}}\, d\theta$$

$$= \frac{GmM}{R} \int_{\pi/4}^{0} \operatorname{sen}\theta\, d\theta = -\frac{GmM}{R}\left[\cos\theta\right]_{\pi/4}^{0}$$

$$= -\frac{GmM}{R}\left(1 - \frac{\sqrt{2}}{2}\right)$$

Además: $W = \cancel{K_0} - K_f$. ($K_0 = 0$)

$$\therefore -\frac{1}{2}\cancel{m}v^2 = -\frac{G\cancel{m}M}{R}\left(1 - \frac{\sqrt{2}}{2}\right)$$

$$v^2 = \frac{GM}{R}\left(2 - \sqrt{2}\right)$$

$$v = \sqrt{\frac{GM}{R}(2 - \sqrt{2})} = \dot{z}$$

$$\therefore \vec{v} = -\sqrt{\frac{GM}{R}(2 - \sqrt{2})}\; \hat{z}$$ ✓

c) $W = \cancel{K_0} - \cancel{K_f} = 0$ ($K_0 = 0$, $K_f = 0$)

$$\therefore W = -\int_{\vec{r}_0}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = 0$$

$$\vec{F} = \left[-\frac{GmMz}{(z^2+R^2)^{3/2}} + \varepsilon F_{nc}(z)\right]\hat{z}$$

$$\Rightarrow -\int_{R}^{0} -\frac{GmMz}{(z^2+R^2)^{3/2}} + \varepsilon F_{nc}(z)\; dz = 0$$

$$\Rightarrow \varepsilon F_{nc}(z) = -\frac{GmMz}{(z^2+R^2)^{3/2}} \quad ; \varepsilon$$

$$F_{nc}(z) = -\frac{1}{\varepsilon}\frac{GmM}{(z^2+R^2)^{3/2}}$$

$$= -\varepsilon \frac{GmM}{(z^2+R^2)^{3/2}}$$