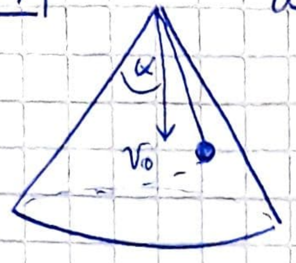
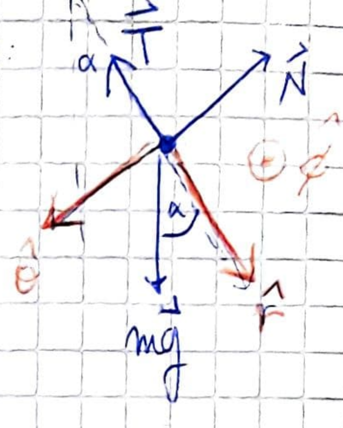
Punto Auxiliar 4 — Berni Ried

P1| a) Usaremos coordenadas esféricas, el origen estará en el vértice del cono. ¡Esto implica que $\theta = cte$!



Si hacemos el DCL obtenemos 3 fuerzas:



Luego:

$$\hat{r}\rfloor \quad mg\cdot\cos\alpha - \vec{T} = m\cdot a_{\hat{r}}$$
$$\hat{\theta}\rfloor \quad mg\,\text{sen}\,\alpha - N = m\cdot\vec{a}_{\hat{\theta}}$$
$$\hat{\phi}\rfloor \quad 0 = m\cdot\vec{a}_{\phi}$$

Del problema notamos que: $\dot{r} = -v_0$; $\theta = \pi - \alpha$
$\ddot{r} = 0$; $\dot{\theta} = \ddot{\theta} = 0$

$$\therefore\ a_{\hat{r}} = (\cancel{\ddot{r}}^{\,0} - \cancel{r\dot{\theta}^2}^{\,0} - r\dot{\phi}^2\sin^2\theta)$$
$$a_{\hat{\theta}} = (\cancel{r\ddot{\theta}}^{\,0} + \cancel{2\dot{r}\dot{\theta}}^{\,0} - r\dot{\phi}^2\sin\theta\cos\theta)$$
$$a_{\hat{\phi}} = (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + \cancel{2r\dot{\theta}\dot{\phi}\cos\theta}^{\,0})$$

Luego las ec. de movimiento quedan:

$$(1)\quad mg\cdot\cos\alpha - T = m(-r\dot{\phi}^2\sin^2\theta)$$
$$(2)\quad mg\,\text{sen}\,\alpha - N = m(-r\dot{\phi}^2\sin\theta\cos\theta)$$
$$(3)\quad 0 = \cancel{m}(r\ddot{\phi}\,\cancel{\sin\theta} - 2v_0\dot{\phi}\,\cancel{\sin\theta})$$

Como $\theta = \pi - \alpha \Rightarrow \text{sen}\,\theta = \text{sen}(\pi - \alpha)$
$= \text{sen}(\pi)\cos(-\alpha) + \text{sen}(-\alpha)\cos(\pi)$
$= \text{sen}(\alpha)$
$\Rightarrow \cos\theta = \cos(\pi - \alpha)$
$= \cos(\pi)\cos(-\alpha) - \text{sen}(\pi)\,\text{sen}(-\alpha)$
$= -\cos(\alpha)$

Notamos también que: $\dot{r} = -v_0 \quad /\int$
$r(t) = -v_0\cdot t + C$
Pero $\rightarrow r(0) = R_0$
$\therefore\ r(t) = R_0 - v_0 t$.

$\therefore$ (1) $\rightarrow mg\cos\alpha - T = -m(R_0 - v_0 t)\dot{\phi}^2\sin^2\alpha$
(2) $\rightarrow mg\sin\alpha - N = +m(R_0 - v_0 t)\dot{\phi}^2\sin\alpha\cos\alpha$
(3) $\rightarrow (R_0 - v_0 t)\ddot{\phi} = 2v_0\dot{\phi}$

$$(3):\ 2v_0\dot{\phi} = (R_0 - v_0 t)\frac{d\dot{\phi}}{dt} \Leftrightarrow \frac{2v_0}{(R_0 - v_0 t)}dt = \frac{d\dot{\phi}}{\dot{\phi}} \quad /\int$$

$$-2\ln(R_0 - v_0 t) + \underset{\hookrightarrow cte}{C} = \ln(\dot{\phi}) \quad / \exp()$$

$$\Leftrightarrow \quad \dot{\phi}(t) = A(R_0 - v_0 t)^{-2} \qquad A = cte \to A(C).$$

Utilizamos la condición inicial $\dot{\phi}(0) = \omega_0$

$$\therefore \dot{\phi}(0) = \frac{A}{R_0^2} = \omega_0 \Rightarrow A = \omega_0 R_0^2$$

$$\therefore \dot{\phi}(t) = \frac{\omega_0 R_0^2}{(R_0 - v_0 t)^2}$$

Luego, en (2): $mg\sin\alpha - N = m\cancel{(R_0 - v_0 t)}\dfrac{\omega_0^2 R_0^4}{(R_0 - v_0 t)^{\cancel{4}\,3}} \cdot \sin\alpha\cos\alpha$

$$\Rightarrow \quad N = m\sin\alpha\left[g - \frac{\omega_0^2 R_0^4 \cos\alpha}{(R_0 - v_0 t)^3}\right]$$

La condición de despegue es: $N \equiv 0$, en t^* ocurre.

$$\therefore g = \frac{\omega_0^2 R_0^4 \cancel{\sin}\cos\alpha}{(R_0 - v_0 t^*)^3} \Rightarrow R_0 - v_0 t^* = \sqrt[3]{\frac{\omega_0^2 R_0^4 \cos\alpha}{g}}$$

$$t^* = \frac{1}{v_0}\left[R_0 - \sqrt[3]{\frac{\omega_0^2 R_0^4 \cos\alpha}{g}}\right] \checkmark$$

$$\therefore r(t^*) = R_0 - v_0 \cdot \frac{1}{v_0}\left[R_0 - \sqrt[3]{\frac{\omega_0^2 R_0^4 \cos\alpha}{g}}\right]$$

$$\therefore \boxed{r(t^*) = \sqrt[3]{\frac{\omega_0^2 R_0^4 \cos\alpha}{g}}} \checkmark$$

b) Despejamos T en (1):

$$T = m\left[g\cos\alpha + \cancel{(R_0 - v_0 t)} \cdot \frac{\omega_0^2 R_0^4}{(R_0 - v_0 t)^{\cancel{4}\,3}} \cdot \sin^2\alpha\right]$$

$$= m\left[g\cos\alpha + \frac{\omega_0^2 R_0^4}{(R_0 - v_0 t)^3}\sin^2\alpha\right]$$

Buscamos $T(t^*)$:

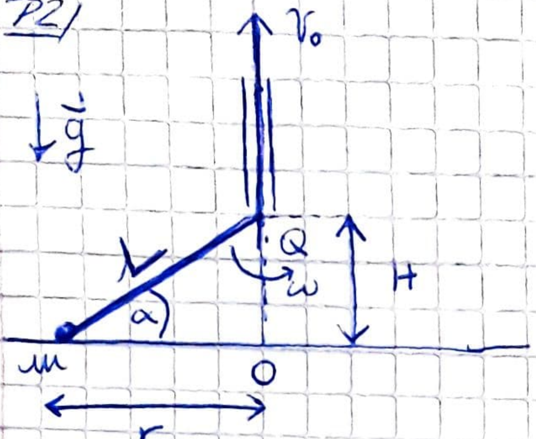
$$T(t^*) = m\left[g\cos\alpha + \frac{\cancel{\omega_0^2 R_0^4} \cdot g}{\cancel{\omega_0^2 R_0^4} \cdot \cos\alpha} \cdot \sin^2\alpha\right]$$

$$= mg\cos\alpha\left[1 + \tan^2(\alpha)\right], \qquad \sin^2\alpha + \cos^2\alpha = 1, \quad \tan^2\alpha + 1 = \frac{1}{\cos^2\alpha}$$

$$= mg\cancel{\cos}\alpha \cdot \frac{1}{\cos\alpha^{2}}$$

$$\boxed{T(t^*) = \frac{mg}{\cos\alpha}} \checkmark \quad \to \text{¡Lo que tiene todo el sentido!} \quad :)$$

P2)



$\dot{L} = -v_0$

En el instante $r = R$, la partícula gira con velocidad angular $\omega = \omega_0$ en torno a OQ.

Se pide determinar $r = r_s$, en que se despega.

a) $L^2 = H^2 + r^2 \quad / \frac{d()}{dt}$

$\Leftrightarrow 2L\dot{L} = 2r\dot{r}$

$\Rightarrow \dot{r} = \frac{L\dot{L}}{r}$, y como $\dot{L} = -v_0 \Rightarrow \dot{r} = -\frac{v_0 L}{r}$

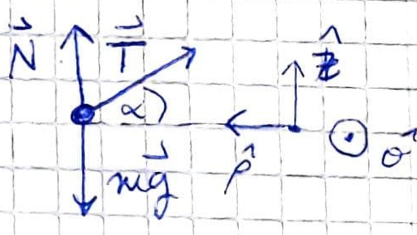
Luego $\ddot{r} = -v_0\left(\frac{\dot{L}r - L\dot{r}}{r^2}\right) = -v_0\left(\frac{-v_0 r - L\cdot\left(-\frac{v_0 L}{r}\right)}{r^2}\right)$

$= \frac{v_0^2}{r^2}\left(r - \frac{L^2}{r}\right)$, y como $L^2 = H^2 + r^2$

$\Rightarrow \ddot{r} = \frac{v_0^2}{r^2}\left(r - \frac{(H^2+r^2)}{r}\right) = \frac{v_0^2}{r^2}\left(\cancel{r} - \cancel{r} - \frac{H^2}{r}\right)$

$$\boxed{\ddot{r} = -\frac{H^2 v_0^2}{r^3}}$$ //

b) Si ocupamos cilíndricas →



$\Rightarrow \hat{\rho}|\ -T\cos\alpha = m\cdot a_{\hat{\rho}}$

$\Rightarrow \hat{z}|\ N - mg + T\operatorname{sen}\alpha = m\cdot a_{\hat{z}}$

$a_{\hat{\rho}} = \ddot{\rho} - \rho\dot{\theta}^2 = \ddot{r} - r\dot{\theta}^2$ y $a_{\hat{z}} = \ddot{z} \overset{!}{=} 0$ en este problema

$\therefore$ $\boxed{\begin{aligned} -T\cos\alpha &= m(\ddot{r} - r\dot{\theta}^2) \\ mg - T\operatorname{sen}\alpha &= N \end{aligned}}$ (1) (2) //

c) Como no hay fuerzas en la dirección $\hat{\theta}$, entonces el momento angular vertical es cte. *

$\therefore\ r^2\dot{\theta} = R^2\omega_0 \Rightarrow \dot{\theta} = \frac{R^2\omega_0}{r^2}$

Luego, (1) queda: $-T\cdot\cos\alpha = m\left(-\frac{H^2 v_0^2}{r^3} - \cancel{r}\cdot\frac{R^4\omega_0^2}{r^{\cancel{4}\,3}}\right)$ //

$T\cos\alpha = \frac{m}{r^3}(H^2 v_0^2 + \omega_0^2 R^4) \Rightarrow \boxed{T(r) = \frac{m}{\cos\alpha\, r^3}(H^2 v_0^2 + \omega_0^2 R^4)}$

$Ec(z) \Rightarrow N = mg - \frac{\sin\alpha}{\cos\alpha} \cdot \frac{m}{r^3}(H^2 v_0^2 + \omega_0^2 R^4)$

d) Imponemos la condición de despegue: $N = 0$; $r = r_s$

$$\Rightarrow g = \frac{\tan\alpha \cdot (H^2 v_0^2 + \omega_0^2 R^4)}{r_s^3}$$

$$r_s^3 = \frac{(H^2 v_0^2 + \omega_0^2 R^4)\tan\alpha}{g} \quad ; \tan\alpha = \frac{H}{r}!$$

$$r_s^4 = \frac{H(H^2 v_0^2 + \omega_0^2 R^4)}{g}$$

$$\boxed{r_s = \sqrt[4]{\frac{H(H^2 v_0^2 + \omega_0^2 R^4)}{g}}}$$

✓✓

* Ya, ¿porqué nos da eso?

→ Como no hay fuerzas en el eje $\hat{\theta}$, el momento angular se conserva. ¿Cuál es el momento angular?

→ $\vec{l} = \vec{r} \times m\vec{v}$; $\vec{r} = r\hat{\rho}$ y $\vec{v} = r\dot{\theta}\hat{\theta} + \dot{r}\hat{\rho}$ (Solo nos movemos en el plano)

$$\Rightarrow \vec{l} = r\hat{\rho} \times m(r\dot{\theta}\hat{\theta} + \dot{r}\hat{\rho})$$
$$= rm(r\dot{\theta}\underbrace{(\hat{\rho} \times \hat{\theta})}_{=\hat{z}} + \dot{r}\underbrace{(\hat{\rho} \times \hat{\rho})}_{=0})$$

$\therefore \vec{l} = mr^2\dot{\theta}\hat{z}$ → Como se conserva:

$$mr^2\dot{\theta}\hat{z} = mR^2\omega_0\hat{z}$$

$$\Rightarrow \dot{\theta} = \frac{R^2\omega_0}{r^2}$$ ✓✓