P1)

a) $\phi_2 = a_2 \frac{x^2}{2} + b_2 xy + \frac{c_2}{2} y^2$

$\phi_4 = \frac{a_4}{12} x^4 + \frac{b_4}{6} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{6} x y^3 + \frac{e_4}{12} y^4$

Para ϕ_2

$$\frac{\partial^4 \phi_2}{\partial x^4} = 0 ; \quad \frac{\partial^4 \phi_2}{\partial^2 x \partial^2 y} = 0 ; \quad \frac{\partial^4 \phi}{\partial y^4} = 0$$

$$\nabla^4 \phi_2 = \frac{\partial^4 \phi_2}{\partial x^4} + 2 \frac{\partial^4 \phi_2}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi_2}{\partial y^4} = 0$$

$\Rightarrow 0 + 2 \cdot 0 + 0 = 0$ //

Para ϕ_4

$$\frac{\partial^4 \phi_4}{\partial x^4} = 2a_4 ; \quad \frac{\partial^4 \phi_4}{\partial x^2 \partial y^2} = 2c_4 ; \quad \frac{\partial^4 \phi_4}{\partial y^4} = 2e_4$$

$$\nabla^4 \phi_4 = \frac{\partial^4 \phi_4}{\partial x^4} + 2 \frac{\partial^4 \phi_4}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi_4}{\partial y^4} = 0$$

$2a_4 + 4c_4 + 2e_4 = 0$

$\Rightarrow \boxed{e_4 = -(a_4 + 2c_4)}$

b) $\sigma_x = \frac{\partial^2 \phi}{\partial y^2} ; \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} , \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$

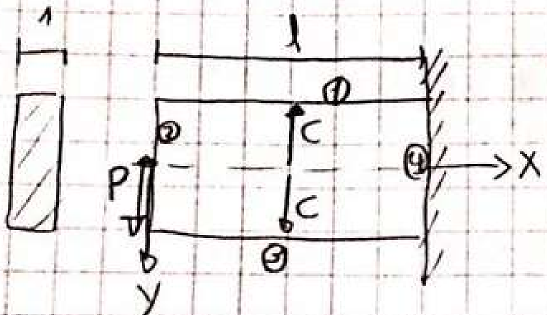
$\phi_2 \Rightarrow \sigma_x = \frac{\partial^2 \phi_2}{\partial y^2} = c_2 ; \quad \sigma_y = \frac{\partial^2 \phi_2}{\partial x^2} = a_2 ; \quad \tau_{xy} = -b_2$

$\phi_4 \Rightarrow \sigma_x = c_4 x^2 + d_4 xy + e_4 y^2 = c_4 x^2 + d_4 xy - (a_4 + 2c_4) y^2$

$$\sigma_x = \sigma_{x(4to\ orden)} = c_4 x^2 + d_4 xy - (a_4 + 2c_4) y^2$$

$$\sigma_y = \sigma_{y(4to\ orden)} = a_4 x^2 + b_4 xy + c_4 y^2$$

$$\tau_{xy} = \tau_{xy(2do\ orden)} + \tau_{xy(4to\ orden)} = -b_4 \frac{x^2}{2} - 2c_4 xy - d_4 \frac{y^2}{2} - b_2$$



$$\partial B_t = ① \cup ② \cup ③$$

$$\partial B_u = ④$$

∂B_t

$$① \Rightarrow \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 0 \leq x \leq l \quad y = -c$$

$$\underset{\sim}{T} \ \underset{\sim}{n} = \underset{\sim}{t}$$

$$\Rightarrow -\sigma_y(x, -c) = 0 \quad (i$$

$$-\tau_{xy}(x, -c) = 0 \quad (ii$$

$$③ \Rightarrow \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \tau_{xy}(x, c) = 0 \quad (iii \\ \sigma_y(x, c) = 0 \quad (iv \end{matrix}$$

$$② \Rightarrow x = 0 \ ; -c \leq y \leq c \qquad \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ t_p \end{pmatrix}$$

$$\sigma_x(0, y) = 0 \quad (v$$

$$-\tau_{xy} = t_p \quad (vi$$

de a)

$$\Rightarrow \sigma_x(0,y) = 0 = -(a_4+2c_4)y^2$$

$$\Rightarrow a_4 = -2c_4$$

$$i) \Rightarrow \sigma_y(x,-c) = 0 = a_4x^2 - b_4xc + c_4c^2$$

$\forall x$, en particular $x=0$

$$\Rightarrow c_4 = 0$$

$$\Rightarrow a_4 = 0$$

$$\Rightarrow 0 = -b_4xc \Rightarrow b_4 = 0$$

$$\Rightarrow \sigma_x = d_4xy$$

$$\sigma_y = 0$$

$$\tau_{xy} = -d_4\frac{y^2}{2} - b_2$$

de ii)

$$\Rightarrow \tau_{xy} = -d_4\frac{c^2}{2} - b_2 = 0$$

$$\Rightarrow b_4 = -d_4\frac{c^2}{2}$$

$$\Rightarrow \tau_{xy} = -\frac{d_4}{2}(y^2-c^2)$$

$\Rightarrow$ de iii)

$$\frac{d_4}{2}(y^2-c^2) = t_p / \int dA \Rightarrow \int_{-c}^{c}\frac{d_4}{2}(y^2-c^2)\,dy = P$$

$$\Rightarrow \frac{d_4}{2}\left(\frac{y^3}{3}\Big|_{-c}^{c} - c^2y\Big|_{-c}^{c}\right) = P$$

$$\Rightarrow \frac{d_4}{2}\left(\frac{2}{3}c^3 - 2c^3\right) = P \Rightarrow d_4 = -\frac{3}{2}\frac{P}{c^3} \Rightarrow \tau_{xy} = \frac{3}{4}\frac{P}{c^3}(y^2-c^2)$$

P2

$$\phi(x,y) = A\left(-4y^5 + 20x^2y^3 - 15x^2yh^2 - 5y^3L^2 + 2y^3h^2 + 5x^2h^3\right)$$

1.- Verificar que cumple con la ecuación biarmónica

$$\frac{\partial^4 \phi}{\partial x^4} + \frac{2\cdot\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$$

$$\frac{\partial \phi}{\partial x} = A\left(20\cdot 2\, x y^3 - 15\cdot 2\cdot x\cdot y\, h^2 + 5\cdot 2\cdot x h^3\right)$$

$$\frac{\partial^2 \phi}{\partial x^2} = A\left(40y^3 - 30yh^2 + 10h^3\right)$$

$$\frac{\partial^3 \phi}{\partial x^2 \partial y} = A\left(40\cdot 3y^2 - 30h^2\right)$$

$$\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = A\left(40\cdot 3\cdot 2\, y\right) = A\cdot 240y$$

$$\frac{\partial^4 \phi}{\partial x^4} = 0$$

$$\frac{\partial^4 \phi}{\partial y^4} = A\left(-4\cdot 5\cdot 4\cdot 3\cdot 2\cdot y\right) = -A\cdot 480\cdot y$$

↳ Reemplazando en la biarmónica

$$0 + 2\cdot A\cdot 240\cdot y - A\cdot 480\cdot y = 0$$

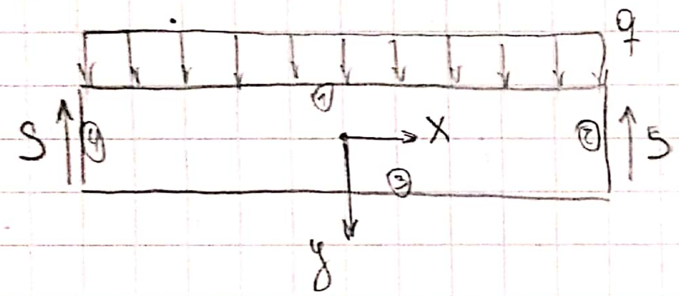
=> cumple.

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = A\left(-80y^3 + 120x^2y - 30yL^2 + 12yh^2\right)$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = A\left(40y^3 - 30yh^2 + 10h^3\right)$$

$$\tau_{xy} = \frac{\partial^2\phi}{\partial x \partial y} = A\left(120xy^2 - 30xh^2\right)$$

Condiciones de borde (Valor de frontera)



$$T = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix}$$

$$\Sigma F = 0 \iff 2S - q \cdot L = 0 \implies \boxed{S = \frac{q \cdot L}{2}}$$

$$\partial B_t = ① \cup ② \cup ③ \cup ④$$

borde ① $-\frac{L}{2} \leq x \leq \frac{L}{2}$, $y = -\frac{h}{2}$, $n = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$T \cdot n = t \iff \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ q \end{pmatrix}$$

$$-\tau_{xy}\left(x, -\tfrac{h}{2}\right) = 0 \implies \text{No aporta}$$

$$-\sigma_y\left(x, -\tfrac{h}{2}\right) = q \implies A = \frac{q}{20h^3} \quad \checkmark\checkmark$$

Se pide el campo de desplazamiento u, para lo que necesitamos ε.

$$\varepsilon_{11} = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}(\sigma_x+\sigma_y)$$

$$= \frac{-2A}{E}\left[5\cdot\nu\cdot(4y^3-3h^2\cdot y+h^3)-(60x^2-40\cdot y^2+6h^2-15L^2)\cdot y\right]$$

$$\varepsilon_{22} = \frac{1+\nu}{E}\sigma_y - \frac{\nu}{E}(\sigma_x+\sigma_y)$$

$$= \frac{-2A}{E}\left[\nu\cdot(60x^2-40y^2+6h^2-15L^2)y-20y^3+15h^2y-5h^3\right]$$

$$\varepsilon_{12} = \frac{1+\nu}{E}\tau_{12}$$

$$= \frac{(1+\nu)A}{E}(120xy^2-30xh^2)$$

Por las ecuaciones de deformación - desplazamiento:

- $\varepsilon_{11} = \frac{\partial u_1(x,y)}{\partial x} \Rightarrow u_1 = \int \varepsilon_{11}(x,y)\,dx$

(A) $\Rightarrow u_1 = \frac{2A}{E}\cdot x\left[20x^2y-5\nu(4y^3-3h^2y+h^3)-y(40y^2-6h^2+15L^2)\right] + C_1(y)$

- $\varepsilon_{22} = \frac{\partial u_2(x,y)}{\partial y} \Rightarrow u_2 = \int \varepsilon_{22}(x,y)\,dy$

(B) $\Rightarrow u_2(x,y) = \frac{A\cdot y}{E}\left[10y^3\cdot(2\nu+1)-3y\{\nu(20x^2+2h^2-5L^2)+5h^2\}+10h^3\right] + C_2(x)$

- $\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1(x,y)}{\partial y}+\frac{\partial u_2(x,y)}{\partial x}\right) = \frac{1+\nu}{E}\cdot A(120xy^2-30xh^2)$

$$\Leftrightarrow \frac{Ax}{E}\left[20x^2-3\{5\nu(8y^2-h^2)+40y^2-2h^2+5L^2\}\right]+\frac{1}{2}\frac{\partial C_1(y)}{\partial y}+\frac{1}{2}\frac{\partial C_2(x)}{\partial x}$$

$$= \frac{1+\nu}{E}\cdot A(120xy^2-30xh^2)$$

$$\frac{A}{E}\cancel{(1+\nu)}\cancel{(120xy^2)} + \frac{A}{E}6h^2x + \frac{A}{E}15\nu h^2x + \frac{A}{E}(20x^3 - 15L^2x)$$ (4)

$$+ \frac{1}{2}\frac{\partial C_1(y)}{\partial y} + \frac{1}{2}\frac{\partial C_2(x)}{\partial x} = \frac{1+\nu}{E}\cdot A(\cancel{120xy^2} - 30xh^2)$$

$$= \frac{-A}{E}30xh^2 - \frac{A}{E}30\nu xh^2$$

$$\Leftrightarrow \frac{1}{2}\left(\frac{\partial C_1(y)}{\partial y} + \frac{\partial C_2(x)}{\partial x}\right) = -\frac{36\cdot A}{E}h^2\cdot x - \frac{45\cdot A}{E}\nu h^2\cdot x - \frac{A}{E}(20x^3 - 15L^2x)$$

$$\Rightarrow \frac{\partial C_1(y)}{\partial y} = 0 \Rightarrow \boxed{C_1(y) = C_3 \to cte}$$

$$\Leftrightarrow \frac{\partial C_2(x)}{\partial x} = -\frac{2A}{E}\cdot(36h^2x + 45\cdot\nu h^2x + 20x^3 - 15L^2x)$$

$$\Rightarrow \boxed{C_2(x) = \frac{-A}{40E}\left[20x^2 + 45h^2\cdot\nu + 36h^2 - 15L^2\right]^2 + C_4}$$

$\Rightarrow$ en (A)

$$u_1(x,y) = \frac{2A}{E}\cdot x\left[20x^2y - 5\nu(4y^3 - 3h^2y + h^3) - y(40y^2 - 6h^2 + 15L^2)\right]$$
$$+ C_3$$

$$u_2(x,y) = \frac{A\cdot y}{E}\left[10y^3(2\nu - 1) - 3y\{\nu(20x^2 + 2h^2 - 5L^2) + 5h^2\} + 10h^3\right]$$
$$- \frac{A}{40E}\left[20x^2 + 45h^2\cdot\nu + 36h^2 - 15L^2\right]^2 + C_4$$

Como está simplemente apoyado, $u_1\left(-\frac{L}{2},\frac{h}{2}\right) \overset{Ⓒ}{=} 0$ y $u_2\left(\frac{L}{2},\frac{h}{2}\right) \overset{Ⓓ}{=} 0$

$$Ⓒ \Rightarrow \boxed{C_3 = -\frac{A}{E}\cdot h(2h^2 + 5L^2)\cdot L}$$

$$\text{(D)} \Rightarrow C_4 = \frac{A}{E}\left[2025\cdot h^4 \nu^2 + 50h^2\left(65h^2 - 18L^2\right)\cdot \nu + 1271\cdot h^4 - 720h^2L^2 + 100L^4\right]$$

$$\therefore \bar{u} = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} \quad ; \quad A = \frac{q}{20h^3}$$

con
$$u_1(x,y) = \frac{q}{10h^3E}\cdot x\left[20x^2y - 5\cdot\nu\left(4y^3 - 3h^2y + h^3\right) - y\left(40y^2 - 6h^2 + 15L^2\right)\right] - \frac{q\cdot L}{20h^2E}\left(2h^2 + 5L^2\right)$$

$$u_2(x,y) = \frac{q\cdot y}{20h^3E}\left[10y^3(2\nu-1) - 3y\left\{\nu\left(20x^2 + 2h^2 - 5L^2\right) + 5h^2\right\} + 10h^3\right] - \frac{q}{800h^3E}\left[20x^2 + 45h^2\cdot\nu + 36h^2 - 15L^2\right]^2 + \frac{q}{20h^3E}\left[2025\cdot h^4\nu^2 + 50h^2\left(65h^2 - 18L^2\right)\cdot\nu + 1271\cdot h^4 - 720h^2L^2 - 100L^4\right]$$