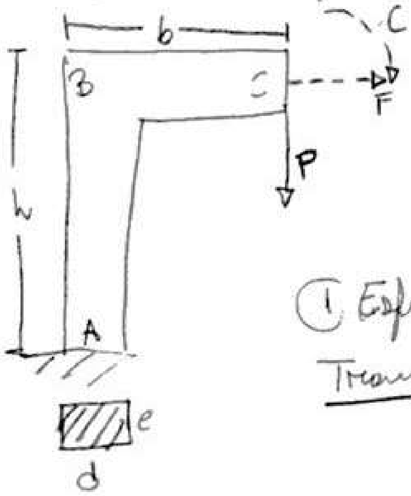
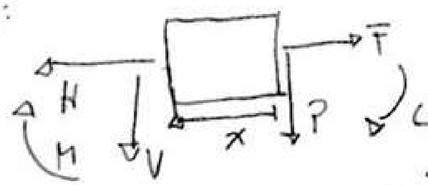
Pauta P1] Aux Castigliano $\delta_y, \delta_x, \theta_c$?



· Como se desea conocer la deflexión en x y el ángulo de rotación, se agregan las fuerzas y momento ficticios $\vec{F}$ y $\vec{C}$ respectivamente. Aplicados en el punto C. F y C se anulan al aplicar Castigliano

① Esfuerzos internos.

Tramo BC:



$\Rightarrow$ Ec de equilibrio

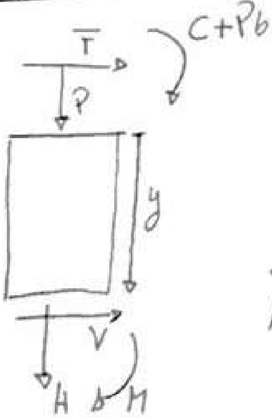
$\sum F_x$: $H = F$ → tracción

$\sum F_y$: $V = -P$ → corte

$\sum M_B$: $M = -C - Px$ → flexión

Tramo AB:

Se suma el torque generado por P



$\sum F_x$: $V = -F$ → corte

$\sum F_y$: $H = -P$ → compresión

$\sum M_B$. $M = -C - Pb - Fy$ → flexión

2) Energía total: cada esfuerzo

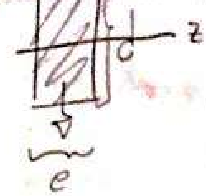
Tracción/Compresión: $U_{tr} = \frac{P^2L}{2EA}$ $\Rightarrow U_{tr} = U^{AB}_{comp} + U^{BC}_{tr} = \frac{P^2h}{2Eed} + \frac{F^2b}{2Eed} = \frac{1}{2Eed}\left(P^2h + F^2b\right)$

ed

Corte $U_{co} = \frac{3}{5}\int_0^L \frac{V^2(x)dx}{GA} \rightarrow U_{co} = U^{AB}_{co} + U^{BC}_{co} = \frac{3}{5}\int_0^h \frac{F^2\,dy}{Ged} + \frac{3}{5}\int_0^b \frac{P^2\,dx}{Ged}$

$= \frac{3}{5Ged}\left(F^2h + P^2b\right)$

Flexión: $U_{fl} = \int_0^L \frac{M^2(x)}{2EI_z}dx \qquad I_z = \frac{ed^3}{12}$



$$= U_{fl}^{AB} + U_{fl}^{BC}$$

$$U_{fl}^{AB} = \int_0^h \frac{(-C-Pb-Fy)^2}{2E\frac{ed^3}{12}}dy = \frac{6}{Eed^3}\int_0^h (+C+Pb+Fy)^2 dy \qquad \text{C.V.} \to u = C+Pb+Fy, \quad du = F\,dy$$

$$= \frac{6}{Eed^3}\int_{C+Pb}^{C+Pb+Fh} \frac{u^2}{F}du = \frac{6}{FEed^3}\left(\frac{(C+Pb+Fh)^3-(C+Pb)^3}{3}\right)$$

$$= \frac{2}{FEed^3}\left(F^3h^3+3F^2h^2(C+Pb)+3Fh(C+Pb)^2+\cancel{(C+Pb)^3}-\cancel{(C+Pb)^3}\right)$$

$$= \frac{2}{\cancel{F}Eed^3}\left(\cancel{F}\left(F^2h^3+3Fh^2(C+Pb)+3h(C+Pb)^2\right)\right)$$

$$= \frac{2(F^2h^3+3Fh^2(C+Pb)+3h(C+Pb)^2)}{Eed^3}$$

$$U_{fl}^{BC} = \int_0^b \frac{(-C-Px)^2}{2EI_z}dx = \frac{2(P^2b^3+3Pb^2C+3bC^2)}{Eed^3}$$

$$U_{fl} = \frac{2}{Eed^3}\left(F^2h^3+3Fh^2(C+Pb)+3h(C+Pb)^2\right)+\frac{2}{Eed^3}\left(P^2b^3+3Pb^2C+3bC^2\right)$$

$$= \frac{2}{Eed^3}\left(F^2h^3+P^2b^3+3Fh^2(C+Pb)+3Pb^2C+3h(C+Pb)^2+3bC^2\right)$$

$$U_T = U_{cmp}^{AB}+U_{tr}^{BC}+U_{cn}^{AB}+U_{cn}^{BC}+U_{fl}^{AB}+U_{fl}^{BC}$$

$$= \frac{P^2h+F^2b}{2Eed}+\frac{3(F^2h+P^2b)}{5Ged}+\frac{2}{Eed^3}\left(F^2h^3+P^2b^3+3Fh^2(C+Pb)+3Pb^2C+3h(C+Pb)^2+3bC^2\right)$$

(3) Castigliano: $\delta_y = \frac{\partial U_T}{\partial P}\Big|_{C=F=0} \qquad \delta_x = \frac{\partial U_T}{\partial F}\Big|_{C=F=0} \qquad \theta_c = \frac{\partial U_T}{\partial C}\Big|_{C=F=0}$

$$\frac{\partial U_T}{\partial P} = \frac{Ph}{Eed}+\frac{6Pb}{5Ged}+\frac{2}{Eed^3}\left(2Pb^3+3Fh^2b+3b^2C+6h(C+Pb)b\right) = \delta_y$$

$$\Rightarrow \frac{\partial U_T}{\partial P}\Big|_{C=F=0} = \frac{Ph}{Eed}+\frac{6Pb}{5Ged}+\frac{2}{Eed^3}\left(2Pb^3+6Phb^2\right) = \frac{4P}{Eed}\left(\frac{h}{4}+\frac{b^3+3b^2h}{d^2}\right)+\frac{6Pb}{5Ged} = \delta_y$$

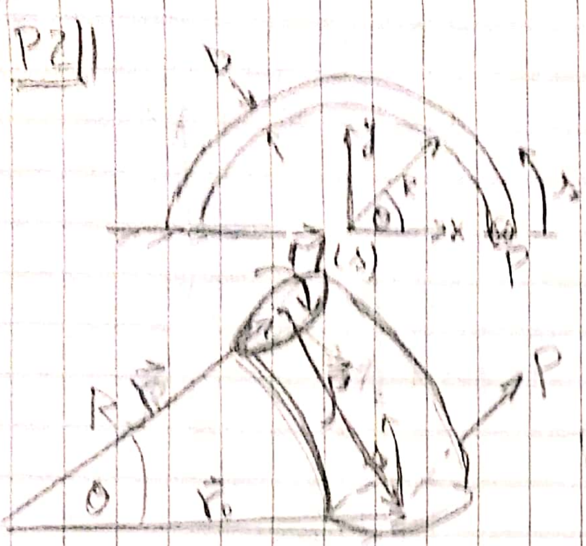
$$\delta_x = \frac{\partial U_T}{\partial F}\bigg|_{\substack{C=0\\F=0}} = \frac{Fb}{Eed} + \frac{6Fh}{5Ged} + \frac{2}{Eed^3}\left(2Fh^3 + 3h^2(C+Pb)\right)$$

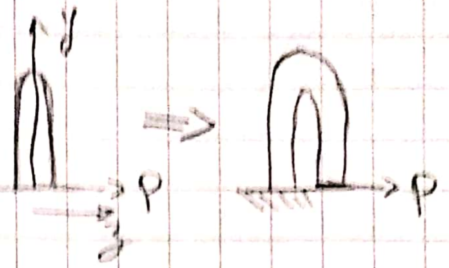
$$= \frac{6Pbh^2}{Eed^3}$$

$$\theta_c = \frac{\partial U_T}{\partial C}\bigg|_{C=F=0} = \frac{2}{Eed^3}\left(3Fh^2 + 3Pb^2 + 6h(C+Pb) + 6bC\right)$$

$$= \frac{2}{Eed^3}\left(3Pb^2 + 6Pbh\right) = \frac{6Pb}{Eed^3}\left(b + 2h\right) \ //$$

P2





* Solo consideraremos flexión y torsión.

$\Rightarrow R^* = R + \frac{D}{2} = 0,48$ [m]

$$\vec{r} = R^*\cos(\theta)\,\hat{i} + R^*\operatorname{sen}(\theta)\,\hat{j} = R^*\cos\left(\frac{s}{R^*}\right)\hat{i} + R^*\operatorname{sen}\left(\frac{s}{R^*}\right)\hat{j}$$

$\vec{M}(s) = \vec{\rho} \times \vec{P}$ $\qquad \vec{P} = -P\,\hat{k}$

$\vec{\rho} = \vec{r}_0 - \vec{r}$ $\quad$ con $\vec{r}_0 = R^*\,\hat{i}$

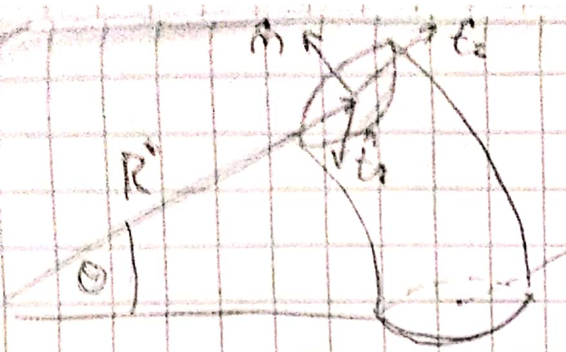
$$\Rightarrow \vec{\rho} = R^*\hat{i} - R^*\cos\left(\frac{s}{R^*}\right)\hat{i} - R^*\operatorname{sen}\left(\frac{s}{R^*}\right)\hat{j}$$

$$\Rightarrow \vec{\rho} = R^*\left(1 - \cos\left(\frac{s}{R^*}\right)\right)\hat{i} - R^*\operatorname{sen}\left(\frac{s}{R^*}\right)\hat{j}$$

$\Sigma M = 0$

$$\Rightarrow \vec{M} = -\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ R^*\left(1-\cos\left(\frac{s}{R^*}\right)\right) & -R^*\operatorname{sen}\left(\frac{s}{R^*}\right) & 0 \\ 0 & 0 & -P \end{vmatrix} = -P\cdot R^*\cdot\operatorname{sen}\left(\frac{s}{R^*}\right)\hat{i} - P\cdot R^*\cdot\left[1-\cos\left(\frac{s}{R^*}\right)\right]\hat{j}$$

↳ Tenemos el momento parametrizado, ahora hay que saber qué componente genera flexión y torsión
→ Para esto, hay que tener un sistema coordenado en la cara donde se mide el momento.



* El producto $\vec{M}\cdot\hat{m}$ generará torsión. Los productos $\vec{M}\cdot\hat{t_1}$ y $\vec{M}\cdot\hat{t_2}$ generarán flexión

$$\hat{m} = \frac{\frac{d\vec{r}}{ds}}{\left\|\frac{d\vec{r}}{ds}\right\|} = \frac{-\sin\left(\frac{s}{R^*}\right)\hat{\imath} + \cos\left(\frac{s}{R^*}\right)\hat{\jmath}}{1} = -\sin\left(\frac{s}{R^*}\right)\hat{\imath} + \cos\left(\frac{s}{R^*}\right)\hat{\jmath}$$

$\vec{t}_1$ es tal que $\vec{t}_1\cdot\hat{m} = 0$

$$\begin{pmatrix} t_{11} \\ t_{12} \\ t_{13} \end{pmatrix} \cdot \begin{pmatrix} -\sin\left(\frac{s}{R^*}\right) \\ \cos\left(\frac{s}{R^*}\right) \\ 0 \end{pmatrix} = -t_{11}\cdot\sin\left(\frac{s}{R^*}\right) + t_{12}\cdot\cos\left(\frac{s}{R^*}\right) = 0$$

$\Rightarrow$ puede ser $t_{11} = \cos\left(\frac{s}{R^*}\right)$ y $t_{12} = \operatorname{sen}\left(\frac{s}{R^*}\right)$

$$\Rightarrow \vec{t}_1 = \hat{t}_1 = \cos\left(\frac{s}{R^*}\right)\hat{\imath} + \operatorname{sen}\left(\frac{s}{R^*}\right)\hat{\jmath}$$

$$\hat{t}_2 = \hat{m}\times\hat{t}_1 = [0 \quad 0 \quad 1] = \hat{k}$$

$$\therefore \text{①}\ M_T = \vec{M}\cdot\hat{m} = -P\cdot R^*\left[\cos\left(\frac{s}{R^*}\right) - 1\right] = T(s)$$

$$\text{②}\ M_{F1} = \vec{M}\cdot\hat{t}_1 = -P\cdot R^*\cdot\sin\left(\frac{s}{R^*}\right) = M(s)$$

$$\text{③}\ M_{F2} = \vec{M}\cdot\hat{t}_2 = 0$$

$$U_{torsión} = \int_0^{\pi R^*} \frac{M_T^2(s)}{2GJ}\,ds = \frac{3\pi P^2 R^{*3}}{4GJ}$$

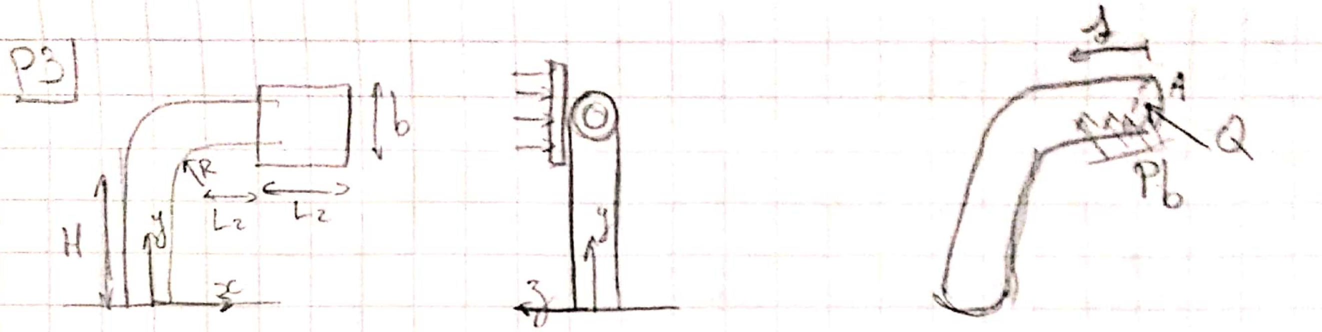
$$U_{flexión} = \int_0^{\pi R^*} \frac{M_{F1}^2(s)}{2EI_z}\,ds = \frac{\pi P^2 R^{*3}}{4EI_z}$$

$$\Rightarrow \boxed{U_{TOT} = \frac{\pi P^2 R^{*3}}{4}\left(\frac{3}{GJ} + \frac{1}{EI_z}\right)}$$

$$\Rightarrow \delta_P = \frac{dU_{TOT}}{dP} = \frac{\pi P R^3}{2}\left(\frac{3}{GJ} + \frac{1}{EI_z}\right)$$

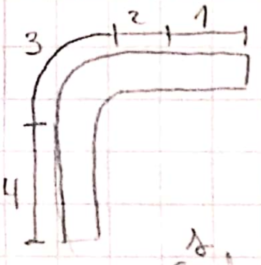
$$\boxed{\begin{aligned} J &= \frac{\pi (R/2)^4}{2} = 2{,}5\cdot 10^{-7} \\ I_z &= \frac{\pi (R/2)^4}{4} = 1{,}3\cdot 10^{-7} \\ \Rightarrow P &= \frac{2\delta_P}{\pi R^3}\left(\frac{1}{GJ} + \frac{1}{EI_z}\right)^{-1} = 544\,[N] \end{aligned}}$$

P3



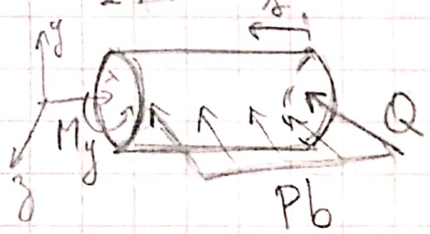
* Se consideró energía por flexión y torsión

4 tramos:



Tramo 1

$0 < s < L_2$

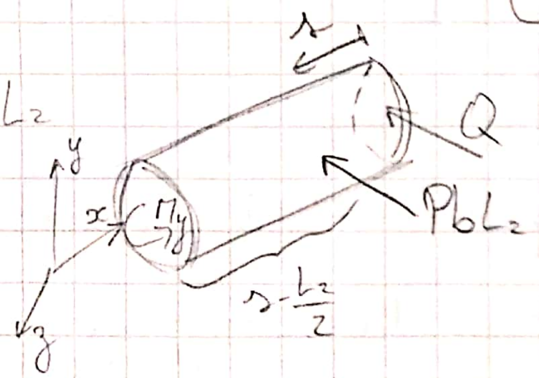


$$\Sigma M : M_y + \frac{Pb \cdot s^2}{2} + Q \cdot s = 0$$

$$\boxed{M_y(s) = -Qs - \frac{Pb\, s^2}{2}}$$

Tramo 2

$L_2 < s < L_1 + L_2$

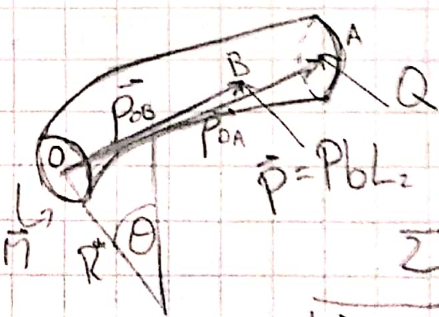


$$\boxed{M_y(s) = -Qs - PbL_2\left(s - \frac{L_2}{2}\right)}$$

Tramo 3

$0 < \theta < \frac{\pi}{2}$

$R^* = R + \frac{D}{2}$



$\vec{Q} = -Q\hat{k}$

$\vec{P} = -PbL_2\hat{k}$

$$\Sigma M_0 : \vec{M} + \vec{\rho}_{OB} \times \vec{P} + \vec{\rho}_{OA} \times \vec{Q} = 0$$

Con:

$$\vec{\rho}_{OA} = \left[R^* \operatorname{sen}(\theta) + L_1 + L_2\right]\hat{\imath} + (1 - \cos(\theta)) \cdot R^* \hat{\jmath} \implies \boxed{\vec{M} = -\vec{\rho}_{OA} \times \vec{Q} - \vec{\rho}_{OB} \times \vec{P}}$$

$$\vec{\rho}_{OB} = \left[R^* \cdot \operatorname{sen}(\theta) + L_1 + \frac{L_2}{2}\right]\hat{\imath} + (1 - \cos(\theta))\, R^* \hat{\jmath}$$

$$-\vec{\rho}_{OA} \times \vec{Q} = -\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ R^* \operatorname{sen}\theta + L_1 + L_2 & R^*(1-\cos(\theta)) & 0 \\ 0 & 0 & -Q \end{vmatrix}$$

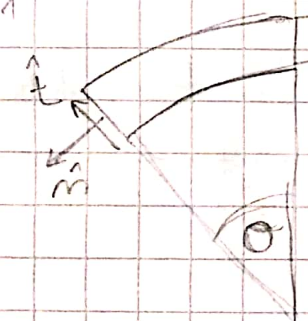
$$= R^* \cdot Q(1-\cos(\theta))\,\hat{\imath} - Q\left(R^* \operatorname{sen}(\theta) + L_1 + L_2\right)\hat{\jmath}$$

$$-\vec{\rho}_{OB} \times \vec{P} = -\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ R^* \operatorname{sen}(\theta) + L_1 + \frac{L_2}{2} & R^*(1-\cos(\theta)) & 0 \\ 0 & 0 & -PbL_2 \end{vmatrix}$$

$$= R^*(Q + PbL_2)(1-\cos(\theta))\,\hat{\imath} - (2Q + PbL_2)\left(R^* \operatorname{sen}(\theta) + L_1 + \frac{L_2}{2}\right)\hat{\jmath}$$

$$\Rightarrow \vec{M}(\theta) = R^*\left[Q(1-\cos(\theta)) + (Q+PbL_2)(1-\cos(\theta))\right]\hat{\imath}$$
$$- \left[Q\left(R^* \operatorname{sen}(\theta) + L_1 + L_2\right) + (2Q + PbL_2)\left(R^* \operatorname{sen}(\theta) + L_1 + \frac{L_2}{2}\right)\right]\hat{\jmath}$$

→ ahora hay que determinar los componentes de torsión y flexión.



$$\hat{m} = -\cos(\theta)\,\hat{\imath} - \operatorname{sen}(\theta)\,\hat{\jmath}$$

$$\hat{t} = \cos(\theta)\,\hat{\jmath} - \operatorname{sen}(\theta)\,\hat{\imath}$$

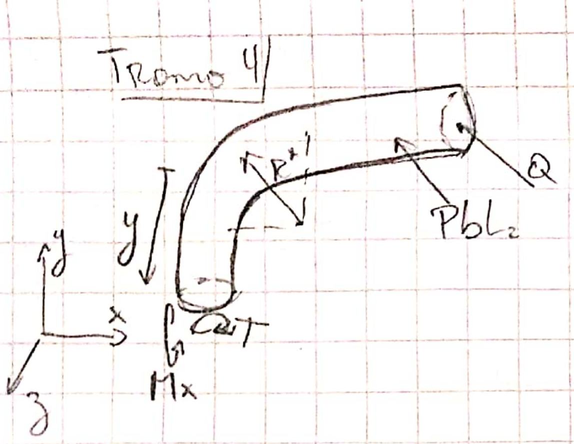
$M_{TAN} = \vec{M} \cdot \hat{t} = M(\theta)$ → componente que causa flexión

$M_{NORM} = \vec{M} \cdot \hat{m} = T(\theta)$ → componente que causa torsión.

$$\Rightarrow M(\theta) = M_{TAN} = -(1-\cos(\theta))\operatorname{sen}(\theta) \cdot R^*(Q + PbL_2)$$
$$- \left[R^* \operatorname{sen}(\theta) + L_1 + \frac{L_2}{2}\right]\cos(\theta) \cdot (2Q + PbL_2)$$

$$T(\theta) = M_{NORM} = -(1-\cos(\theta)) \cdot \cos(\theta) \cdot R^*(Q + PbL_2)$$
$$+ \left[R^* \operatorname{sen}(\theta) + L_1 + \frac{L_2}{2}\right]\operatorname{sen}(\theta)(2Q + PbL_2)$$

Tramo 4/



$$\sum M_y: \left(R^* + L_1 + \frac{L_2}{2}\right) PbL_2 + (R^* + L_1 + L_2)\, Q + T = 0$$

$$\Rightarrow T(y) = -\left(R^* + L_1 + \frac{L_2}{2}\right) PbL_2 - (R^* + L_1 + L_2) \cdot Q$$

$$\sum M_x: M_x - (R^* + y)(PbL_2 + Q) = 0$$

$$\Rightarrow \boxed{M_x(y) = (R^* + y)(PbL_2 + Q)}$$

Con todos los expresiones se puede calcular la energía total.

$$U_{TOT} = \int_0^{L_2} \frac{M_y^2(s)}{2EI_z} ds + \int_{L_2}^{L_1+L_2} \frac{M_y^2(s)}{2EI_z} ds + \int_0^{\pi/2} \frac{M^2(\theta)}{2EI_z} \cdot R^* d\theta + \int_0^{\pi/2} \frac{T^2(\theta)}{2GJ} R^* d\theta$$

$$+ \int_0^{H} \frac{M_x^2(y)}{2EI_z} dy + \int_0^{H} \frac{T^2(y)}{2GJ} dy$$

→ se puede hacer con TI o a mano, aquí lo haremos a ✋

Usando el teorema de Castigliano:

$$\delta_{Az} = \frac{\partial U_{TOT}}{\partial Q}\Big|_{Q=0}$$

* Para resolver la integral usaremos el hecho de que:

$$\frac{d F^2(x)}{dQ} = 2 F(x) \frac{d F(x)}{dQ}$$ (regla de la cadena

entonces nos interesa:

- $0 < s < L_2$: $\dfrac{dM_y}{dQ} = -x$

- $L_2 < s < L_1 + L_2$: $\dfrac{dM_y}{dQ} = -x$

• $0 < \theta < \frac{\pi}{2}$: $\quad \dfrac{dM}{dQ} = -(1-\cos(\theta))\,\mathrm{sen}(\theta)\,R^* - 2\left[R^*\mathrm{sen}(\theta) + L_1 + \frac{L_2}{2}\right]\cdot\cos(\theta)$

$$\frac{dT}{dQ} = -(1-\cos(\theta))\cos(\theta)\,R^* + 2\left[R^*\mathrm{sen}(\theta) + L_1 + \frac{L_2}{2}\right]\mathrm{sen}(\theta)$$

• $0 < y < H$: $\quad \dfrac{dM_x}{dQ} = R^* + y \qquad \dfrac{dT}{dQ} = -(R^* + L_1 + L_2)$

$$\delta_{Ay} = \frac{1}{EI_z}\left\{\int_0^{L_2} M_y(s)\frac{\partial M_y(s)}{\partial Q}\,ds + \int_{L_2}^{L_1+L_2} M_y(s)\frac{\partial M_y}{\partial Q}\,ds + \int_0^{\pi/2} M(\theta)\frac{\partial M}{\partial Q}R^*\,d\theta\right.$$

$$\left. + \int_0^H M_x(y)\frac{\partial M_x(y)}{\partial Q}\,dy\right\} + \frac{1}{GJ}\left\{\int_0^{\pi/2} T(\theta)\frac{\partial T}{\partial Q}\cdot R^*\,\partial\theta + \int_0^H T(y)\frac{\partial T}{\partial Q}\,dy\right\}$$

$$= \frac{1}{EI_z}\left\{\int_0^{L_2}(Qs + Pbs^2)\,s\,ds + \int_{L_2}^{L_1+L_2}\left[Qs + PbL_2\left(s - \frac{L_2}{2}\right)\right]s\,ds\right.$$

$$+ \int_0^{\pi/2}\left\{(1-\cos(\theta))\mathrm{sen}(\theta)R^*(Q + PbL_2) + \left(R^*\mathrm{sen}(\theta) + L_1 + \frac{L_2}{2}\right)\cos(\theta)\cdot(2Q + PbL_2)\right.$$

$$\left. + \left(R^*\mathrm{sen}(\theta) + L_1 + \frac{L_2}{2}\right)\cos(\theta)(2Q + PbL_2)\right\}\left\{(1-\cos(\theta))\mathrm{sen}(\theta)R^*\right.$$

$$\left.\left. + 2\left(R^*\mathrm{sen}(\theta) + L_1 + \frac{L_2}{2}\right)\cos(\theta)\right\}R^*\,d\theta + \int_0^H (R^*+y)(PbL_2 + Q)(R^*+y)\,dy\right\}$$

$$+ \frac{1}{GJ}\left\{\int_0^{\pi/2}\left[-(1-\cos(\theta))\cos(\theta)R^*\cdot(Q + PbL_2) + \left(R^*\mathrm{sen}(\theta) + L_1 + \frac{L_2}{2}\right)\mathrm{sen}(\theta)\right.\right.$$

$$\left.\cdot(2Q + PbL_2)\right]\cdot\left[-(1-\cos(\theta))\cos(\theta)R^* + 2\left(R^*\mathrm{sen}(\theta) + L_1 + \frac{L_2}{2}\right)\mathrm{sen}(\theta)\right]R^*d\theta$$

$$\left. + \int_0^H\left[\left(R^* + L_1 + \frac{L_2}{2}\right)PbL_2 + (R^* + L_1 + L_2)Q\right](R^* + L_1 + L_2)\,dy\right\}$$

Además

$$I = \frac{\pi}{64}\left[D^4 - (D-2e)^4\right] = 8{,}0576 \cdot 10^{-6}\ [m^4]$$

$$J = \frac{\pi}{32}\left[D^4 - (D-2e)^4\right] = 2I$$

$$\Rightarrow \delta_{Az} = 0{,}00612 + 0{,}000052 \cdot Q^{=0}$$

$$\Rightarrow \boxed{\delta_{Az} = 6{,}12\ [mm]}$$