Pauta aux 11

P1)

Sea $T: M_{2,2}(\mathbb{R}) \to \mathbb{R}^4$ definida por

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a-b, b-c, c-d, d-a)$$

a) Pruebe que T es lineal

Sol)

Sea $\lambda \in \mathbb{R}$, $A, B \in M_{2,2}(\mathbb{R})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

$$T(\lambda A + B) = T\begin{pmatrix} \lambda a + x & \lambda b + y \\ \lambda c + z & \lambda d + w \end{pmatrix}$$

$$= (\lambda a + x - [\lambda b + y], \lambda b + y - [\lambda c + z],$$

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$\lambda c + z - [\lambda d + w], \lambda d + w - [\lambda a + x])$

$= (\lambda(a-b) + x - y, \lambda(b-c) + y - z, \lambda(c-d) + z - w,$
$\lambda(d-a) + w - x)$

$= (\lambda(a-b), \lambda(b-c), \lambda(c-d), \lambda(d-a))$
$+ (x-y, y-z, z-w, w-x)$

$= \lambda(a-b, b-c, c-d, d-a)$
$+ (x-y, y-z, z-w, w-x)$

$= \lambda T\begin{pmatrix} a & b \\ c & d \end{pmatrix} + T\begin{pmatrix} x & y \\ z & w \end{pmatrix}$

$\therefore$ T <u>es</u> lineal //

b) Calcule una base y la dimensión para los sev $Ker(T)$ e $Im(T)$

Sol

estudiemos el $Ker(T)$

$$Ker(T) = \{A \in M_{2,2}(\mathbb{R}) : T(A) = 0\}$$

luego $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$T(A) = (a-b, b-c, c-d, d-a) = 0$$

$$\Rightarrow \begin{array}{l} a = b \\ b = c \\ c = d \\ d = a \end{array} \qquad Ker(T) = \left\{a\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : a \in \mathbb{R}\right\}$$

es decir

$$\ker(T) = \left\langle \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \right\rangle$$ base ker

$\therefore \quad \dim(\ker(T)) = 1$

estudiemos la imagen

1- busquemos una base de $M_{2,2}(\mathbb{R})$

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\Rightarrow T(B) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

ahora como B genera el espacio $T(B)$ va a generar la imagen. ahora

Por el TNI

$$\dim(M_{2,2}) = \dim(\ker(T)) + \dim(Im(T))$$

$$4 = 1 + \dim(Im(T))$$

$$\Rightarrow \dim(Im(T)) = 3$$

luego hay que quitar uno delos vectores generadores

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notamos

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow Im(T) = \left\langle \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\} \right\rangle$$

y sabemos que $dim(Im(T)) = 3$ //

c) Estudie la inyectividad y sobreyectividad de T.

Sol : T es inyectiva ssi $dim(Ker(T)) = 0$

Como $\dim(\ker(T)) = 1$

T <u>no</u> es inyectiva

T es epiyectiva

ssi

$\dim(\operatorname{Im}(T)) = \dim(\mathbb{R}^4) = 4$

Como

$\dim(\operatorname{Im}(T)) = 3$

$\Rightarrow$ T <u>no</u> es epiyectiva //

d) considere las bases respectivas de $M_{22}(\mathbb{R})$ y $\mathbb{R}^4$ dadas por:

$$B=\left\{\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\-1&1\end{pmatrix}\right\}$$

y

$$C=\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\-1\\0\\0\end{pmatrix},\begin{pmatrix}1\\-1\\-1\\0\end{pmatrix},\begin{pmatrix}1\\-1\\0\\-1\end{pmatrix}\right\}$$

calcule la matriz representante con respecto a las bases B y C

Sol

Esta es una de las tantas formas de resolver el problema.

1- Calculamos la imagen de la base del espacio de salida

$$T\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad T\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 0 \end{pmatrix}$$

$$T\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$T\left(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

ahora buscamos

(a, b, c, d) ∈ ℝ

$$T(\cdot) = a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} + c\begin{pmatrix}1\\-1\\-1\\0\end{pmatrix} + d\begin{pmatrix}1\\-1\\0\\-1\end{pmatrix}$$

donde "$\cdot$" son los elementos en B

así

$$T\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

$\therefore$

$$\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} = a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} + c\begin{pmatrix}1\\-1\\-1\\0\end{pmatrix} + d\begin{pmatrix}1\\-1\\0\\-1\end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} a+b+c+d \\ -b-c-d \\ -c \\ -d \end{pmatrix}$$

$$\to \quad \begin{aligned} d &= 1 \\ c &= 0 \\ b &= -2 \\ a &= 1 \end{aligned}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a+b+c+d \\ -b-c-d \\ -c \\ -d \end{pmatrix}$$

$d = 0$

$c = -1$

$b = 1$

$a = -1$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b+c+d \\ -b-c-d \\ -c \\ -d \end{pmatrix}$$

$d = -1$

$c = 0$

$b = 2$

$a = -1$

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$f\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} a+b+c+d \\ -b-c-d \\ -c \\ -d \end{pmatrix}$$

$d = 0$

$c = 2$

$b = -4$

$a = 2$

$$\begin{pmatrix} 0 \\ 2 \\ -2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (-4)\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2 \\ -2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & -1 & -1 & 2 \\ -2 & 1 & 2 & -4 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

P2

a) Sea $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ una función lineal que satisface

$$T\begin{pmatrix}1\\1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\quad T\begin{pmatrix}1\\0\\1\\0\end{pmatrix}=\begin{pmatrix}2\\1\\2\\-1\end{pmatrix}$$

$$T\begin{pmatrix}1\\-2\\1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\quad T\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

1) Encuentre la transformación T y su matriz representante

Sol

Para esto se utiliza la siguiente idea

1- Si T es lineal

2- T es conocida en la base del espacio de salida

entonces T es conocida

ie $T: U \to V$, U tiene $\{u_i\}_{i=1}^{n}$ como base y $T(u_i) = v_i \ \forall i$

$\Rightarrow$ T es Totalmente conocida

Sol

Sea $u \in U \Rightarrow u = \sum_{i=1}^{n} \alpha_i v_i$

$$\Rightarrow T(v) = T\left(\sum_{i=1}^{n} \alpha_i v_i\right)$$

Lineal $\Downarrow$

$$= \sum_{i=1}^{n} \alpha_i T(v_i)$$

asi solo basta conocer los $T(v_i)$

ya que los α_i son cualquier

escalar //

luego veamos si

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

son li

$$\alpha\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+\beta\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+\gamma\begin{pmatrix}1\\-2\\1\\1\end{pmatrix}+w\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}=0$$

$$\begin{pmatrix}1&1&1&1\\1&0&-2&1\\0&1&1&0\\0&0&1&-1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\\w\end{pmatrix}=0$$

veamos si la matriz
es invertible

$$\begin{pmatrix}1&1&1&1\\1&0&-2&1\\0&1&1&0\\0&0&1&-1\end{pmatrix}$$

$f_2' = f_2 - f_1$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\downarrow \; f_3' = f_3 + f_2$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\downarrow \; f_4' = f_4 + \frac{1}{2} f_3$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

todos son distintos de cero

$\therefore$ es invertible

y son li

luego como $\dim(\mathbb{R}^4)=4$

$\Rightarrow$ son base

luego dado

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$ busquemos

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix} + W \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ w \end{pmatrix}$$

busquemos la inversa

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right)$$

$\downarrow$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -1 & -3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right)$$

$\downarrow$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & -2 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right)$$

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$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & 1 \\ 0 & -1 & 0 & 0 & 1/2 & -1/2 & -3/2 & 0 \\ 0 & 0 & -2 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1/2 & 1/2 & 1/2 & 1 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1/2 & 1 \\ -1/2 & 1/2 & 3/2 & 0 \\ 1/2 & -1/2 & -1/2 & 0 \\ 1/2 & -1/2 & -1/2 & 1 \end{pmatrix}$$

luego

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ w \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1/2 & 1 \\ -1/2 & 1/2 & 3/2 & 0 \\ 1/2 & -1/2 & -1/2 & 0 \\ 1/2 & -1/2 & -1/2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a+b-c+2d}{2} \\ \frac{-a+b+3c}{2} \\ \frac{a-b-c}{2} \\ \frac{a-b-c-2d}{2} \end{pmatrix}$$

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = T\left(\alpha\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right) + T\left(\beta\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right)$$

$$+ T\left(\gamma\begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}\right) + T\left(w\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right)$$

$$= \alpha T\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta T\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma T\begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

$$+ wT\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \alpha\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix} + \gamma\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$+ W\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

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$$= \frac{a+b-c+2d}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$+ \frac{-a+b+3c}{2}\begin{pmatrix}2\\1\\2\\-1\end{pmatrix}$$

$$+ \frac{a-b-c}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$+ \frac{a-b-c-2d}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

$$= a\left(\frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}2\\1\\2\\-1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right)$$

$$+ b\left(\frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2\\1\\2\\-1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right)$$

$$+ c\left[-\frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+\frac{3}{2}\begin{pmatrix}2\\1\\2\\-1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right]$$

$$+ d\left[\begin{pmatrix}1\\0\\1\\0\end{pmatrix}-\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right]$$

$$= a\begin{pmatrix}1/2\\0\\-1/2\\1\end{pmatrix}+b\begin{pmatrix}1/2\\0\\3/2\\-1\end{pmatrix}$$

$$+ c\begin{pmatrix}3/2\\1\\5/2\\-2\end{pmatrix}+d\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix} \frac{a+b+3c}{2} \\ c-d \\ \frac{-a+3b+5c+2d}{2} \\ a-b-2c \end{pmatrix} //$$

2) de una base del ker de T

$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \ker(T)$

ssi

$$\frac{a+b+3c}{2} = 0$$

$$c-d = 0$$

$$\frac{-a+3b+5c+2d}{2} = 0$$

$$a-b-2c = 0$$

ssi

$$a + b + 3c = 0$$

$$c - d = 0$$

$$-a + 3b + 5c + 2d = 0$$

$$a - b - 2c = 0$$

ssi

$$\begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 3 & 5 & 2 \\ 1 & -1 & -2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

escalonamdo

$$\begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 3 & 5 & 2 \\ 1 & -1 & -2 & 0 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 4 & 8 & 2 \\ 0 & -2 & -5 & 0 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 4 & 8 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \quad \begin{aligned} a + b + 3c &= 0 \\ -2b - 5c &= 0 \\ -2c + 2d &= 0 \end{aligned}$$

$$\rightarrow c = d$$

$c = d$

$a + b + 3d = 0$

$-2b - 5d = 0$

$\rightarrow b = -\frac{5}{2} d$

$\rightarrow a = \frac{5}{2} d - 3d = -\frac{d}{2}$

$$\therefore \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = d \begin{pmatrix} -1/2 \\ -5/2 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \ker(f) = \left\langle \left\{ \begin{pmatrix} -1/2 \\ -5/2 \\ 1 \\ 1 \end{pmatrix} \right\} \right\rangle$$

3) Encuentre una base de $Im(T)$

como $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

generan $\mathbb{R}^4$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

generan la imagen para T N I

$$\dim(\mathbb{R}^4) = \dim(Ker(T)) + \dim(Im(T))$$

$\Rightarrow \dim(Im(T)) = 3$

$\Rightarrow$ basta encontrar

3 l.i. de

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

son li, son 3 luego

son base

b) Sea V un e.v de dimensión finita sobre $\mathbb{R}$ y sea $T: V \to V$ una transformación lineal t.q $T \circ T = T$

pdq $V = Ker(T) \oplus Im(T)$

Sol)

veamos que suman

notar

$\dim(Ker(T) + Im(T))$

$= \dim(Ker(T)) + \dim(Im(T)) - \dim(Ker(T) \cap Im(T))$

$\underset{\substack{\downarrow \\ TNI}}{=} \dim(V) - \dim(Ker(T) \cap Im(T))$

luego si

$\ker(T) \cap \operatorname{Im}(T) = \{0\}$ $(*)$

$\Rightarrow \dim(\ker(T) + \operatorname{Im}(T)) = \dim(V)$

$\Rightarrow \ker(T) + \operatorname{Im}(T) = V$

$\Rightarrow \ker(T) \oplus \operatorname{Im}(T) = V$ //

por $(*)$

mostremos $*$

sea $x \in \ker(T) \cap \operatorname{Im}(T)$

$\Rightarrow T(x) = 0$ y $\exists \overline{x}$ t.q

$T(\overline{x}) = x$

$T(x) = 0$

$\Rightarrow T(T(\overline{x})) = 0$

$\hookrightarrow x = T(\overline{x})$

$\Rightarrow T(\overline{x}) = 0$

$\hookrightarrow T \circ T = T$

$\vdots$

$\Rightarrow x = 0$

$\hookrightarrow T(\overline{x}) = x$

$\therefore \ker(T) \cap \operatorname{Im}(T) = \{0\}$

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P3

a) U ev de dimensión finita y $T: U \to U$ Transformación lineal pdq

1) $T \circ T = 0$ ssi $Im(T) \subseteq Ker(T)$

Sol

=>) sea $Y \in Im(T)$

$\Rightarrow \exists x$ t.q $T(x) = Y$

$\Rightarrow T(T(x)) = T(Y)$

$\underset{0}{\parallel} \to$ hip

$\Rightarrow T(Y) = 0 \Rightarrow Y \in Ker(T)$

$\therefore Im(T) \subseteq Ker(T)$ //

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$\Leftarrow$) PDQ $T(T(x))=0$ $\forall x$

Sea $x \in U$

$\Rightarrow T(x) \in Im(T) \subseteq Ker(T)$

$\Rightarrow T(T(x))=0$

$\therefore T(T(x))=0$ $\forall x \in U$

$\Rightarrow T \circ T = 0$

2) $Im(T) = Ker(T)$

ssi

$\dim(U) = 2\dim(Ker(T))$

y $Im(T) \subseteq Ker(T)$

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$\Rightarrow]$

si $Im(T) = Ker(T)$

$\Rightarrow Im(T) \subseteq Ker(T)$ ✓

y por TMI

$$\dim(V) = \dim(Im(T)) + \dim(Ker(T))$$

$$= \dim(Ker(T)) + \dim(Ker(T))$$

$$= 2\dim(Ker(T))$$

$\Leftarrow]$ por TMI

$$\dim(V) = \dim(Im(T)) + \dim(Ker(T))$$

$\downarrow \to$ hip

$$2\dim(Ker(T))$$

$\Rightarrow \dim(Im(T)) = \dim(Ker(T))$

y $Im(T) \subseteq Ker(T)$

$\Rightarrow Im(T) = Ker(T)$ //

b) Encontrar todas las

transformaciones lineales

$T: \mathbb{R}^2 \to \mathbb{R}^2$ t.q

$Ker(T) = \langle \{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \} \rangle$

y $Ker(T) = Im(T)$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ generan $\mathbb{R}^2$

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$$\Rightarrow T\begin{pmatrix} a \\ b \end{pmatrix} = T\begin{pmatrix} a \\ 0 \end{pmatrix} + T\begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$= aT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cancel{bT\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{0}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$= aT\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

pan

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in Ker(T)$

ahora $\quad Im(T) = \langle \{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \} \rangle$

$$\Rightarrow aT\begin{pmatrix} 1 \\ 0 \end{pmatrix} = b\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \forall a \in \mathbb{R}$$

si $a = 1$ $\exists \hat{b}$ t.q.

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{b}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

luego

$$T\begin{pmatrix} a \\ b \end{pmatrix} = a\hat{b}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{b}\begin{pmatrix} 0 \\ a \end{pmatrix}$$

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P4)

Sea $MS_{2,2}(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ como:

$$T\begin{pmatrix} a & b \\ b & c \end{pmatrix}(x) = c + (b+a)x + 2ax^2$$

considere además las bases

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

y

$$\beta' = \{1, x, x^2\}$$

a) demuestre que T es lineal

Sol

$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, $B = \begin{pmatrix} d & e \\ e & f \end{pmatrix}$, $\lambda \in \mathbb{R}$

p.d. $T(\lambda A + B) = \lambda T(A) + T(B)$

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$$T(\lambda A+B)(x)$$

$$=T\begin{pmatrix} \lambda a+d & \lambda b+e \\ \lambda b+e & \lambda c+f \end{pmatrix}(x)$$

$$=\lambda c+f+[\lambda a+d+\lambda b+e]x$$
$$+2[\lambda a+d]x^2$$

$$=\lambda c+[\lambda a+\lambda b]x+2\lambda a x^2$$
$$+f+(d+e)x+2dx^2$$

$$=\lambda[c+(a+b)x+2ax^2]$$
$$+f+(d+e)x+2dx^2$$

$$=\lambda T(A)+T(B)$$

b) ¿T puede ser epiyectiva?

¿T puede ser inyectiva?

Sol

observar que

$\dim(MS_{2\times2}(\mathbb{R}))=4$ } distintos

$\dim(P_2(\mathbb{R}))=3$

$\therefore$ T no puede ser inyectiva

y T podría ser epiyectiva

no nos piden probarlo

solo decir si puede.

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c)

Calcular la matriz A con respecto a las bases β y β'

Sol

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 + (0+1)x + 2\cdot 1\cdot x^2$$

$$= x + 2x^2$$

$$T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 + (1+0)x + 2\cdot 0 \cdot x^2$$

$$= x$$

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 + (0+0)x + 2\cdot 0\cdot x^2$$

$$= 1$$

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$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 + x + 2x^2$$

$$T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 + x + 0x^2$$

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 + 0x + 0x^2$$

$$\Rightarrow A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

luego

$$T\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

d) Encuentre $S: MS_{2,2}(\mathbb{R}) \to P_2(\mathbb{R})$ cuya matriz representante con respecto a las bases β y β'

$$\tilde{T}\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\tilde{T}\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} b+2c \\ b \\ a \end{pmatrix} \begin{matrix} 1 \\ x \\ x^2 \end{matrix}$$

$$\Rightarrow \tilde{T}\begin{pmatrix} a & b \\ b & c \end{pmatrix}(x) = b+2c+bx+ax^2$$

P5/ Sea V un ev con Base $B = \{u, v, w\}$ y sea $T: V \to V$ una transformación lineal t.q

$T(u) = v - w$

$T(v) = u + v$

$T(w) = u + 2v - w$

a) Determine la matriz representante de T con respecto a las bases B y B

Sol)

$$T(u) = 0 + v - w$$

$$T(v) = u + v + 0$$

$$T(w) = u + 2v - w$$

$$\Rightarrow T(au + bv + cw) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

entendiendo $u \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$v \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$w \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

b) Sea L el sev generado por $2u+3v-w$

Encuentre los siguientes sev y sus dimensiones

1) $T(L)$

Sol

Como $L = \langle \{ 2u+3v-w \} \rangle$

basta estudiar

$T(2u+3v-w)$

$$\rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 \\ 2+3 & -2 \\ -2 & +1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

luego $T(2u+3v-w) = 2u+3v-w$

$\Rightarrow T(L) = L$

$\dim(T(L)) = 1$

y $\{2u+3v-w\}$ es base //

2) $L \cap \ker(T)$

Sol

Veamos: $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

$$b + c = 0$$
$$a + b + 2c = 0$$
$$-a - c = 0$$

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$\rightarrow \quad a = -c$

$b + c = 0 \rightarrow \quad b = -c$

$a + b + 2c = 0$

$\rightarrow \quad a = b = -c$

$a + 2b + 2c = 0$

$\rightarrow$ luego $\left\langle \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\} \right\rangle = \mathrm{Ker}(t)$

luego veamos si $\exists\, a$ / $\underline{q}$

$a \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \in L$

$$a\begin{pmatrix}1\\1\\-1\end{pmatrix} = b\begin{pmatrix}2\\3\\1\end{pmatrix}$$

Claramente solo el cero cumple

$\Rightarrow L \cap \ker(t) = \{0\}$

$\therefore \dim(L \cap \ker(t)) = 0$

3) $\ker(t) = \left\langle \left\{ \begin{pmatrix}1\\1\\-1\end{pmatrix} \right\} \right\rangle$

$$L = \left\langle \left\{ \begin{pmatrix}2\\3\\-1\end{pmatrix} \right\} \right\rangle$$

$\ker(t) + L = \left\langle \left\{ \begin{pmatrix}2\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix} \right\} \right\rangle$

son li $\therefore$

$\dim(\ker(t) + L) = 2$

4)

$Im(T)$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= a\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\Rightarrow Im(T) = \left\langle \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\} \right\rangle$$

veamos si son li

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ son l.i}$$

$$\Rightarrow \dim\left(\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \right) = 2$$

P6]

Sean V, W ev reales y $T: V \rightarrow W$ una función lineal. Sea $\{u_1, \ldots, u_k\} \subseteq V$ un conjunto l.i

pdq

$\{T(u_1), \ldots, T(u_n)\}$ son l.i

ssi $\langle \{u_1, \ldots, u_n\} \rangle \cap \ker(T) = \{0\}$

Sol

$\Rightarrow$)

sea $x \in \langle \{u_1, \ldots, u_n\} \rangle \cap \ker(T)$

$$\Rightarrow x = \sum_{i=1}^{n} a_i u_i$$

$$\Rightarrow T(x) = 0$$

$$\Rightarrow T\left(\sum_{i=1}^{n} a_i u_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} a_i T(u_i) = 0$$

$\Rightarrow a_i = 0 \ \forall i$ por l.i

$\Rightarrow x = 0$

$\Rightarrow \langle \{u_1, \ldots, u_n\} \rangle \cap \ker(T) = \{0\}$

$\boxed{\subseteq}$

sea $\{T(u_i)\}_{i=1}^{n}$ veamos

si son li

$$\sum_{i=1}^{n} a_i T(u_i) = 0$$

$$\Rightarrow T\left(\sum_{i=1}^{n} a_i u_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} a_i u_i \in \ker(T) \cap \langle \{u_1, \ldots, u_n\} \rangle$$

$$\Rightarrow \sum_{i=1}^{n} a_i u_i = 0 \Rightarrow a_i = 0 \ \forall i$$

$\hookrightarrow \{u_i\}$ li

$\Rightarrow \langle \{v_1, \ldots, v_m\} \rangle \cap \ker(T) = \{0\}$

$\boxed{\Leftarrow}$

sea $\{T(v_i)\}_{i=1}^{m}$ veamos

si son li

$$\sum_{i=1}^{m} a_i T(v_i) = 0$$

$$\Rightarrow T\left(\sum_{i=1}^{m} a_i v_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{m} a_i v_i \in \ker(T) \cap \langle \{v_1, \ldots, v_m\} \rangle$$

$$\Rightarrow \sum_{i=1}^{m} a_i v_i = 0 \Rightarrow a_i = 0 \ \forall i$$

$\hookrightarrow \{v_i\}$ li

$\Rightarrow$ Son l.i. //

P7) Sea $L: P_3 \to P_3$ definida por

$L[p](x) = p(0)x^3 + p(1)(x-4)^2$

a) Pruebe que L es una transformación lineal

Sol:

Sea $p, q \in P_3$, $\lambda \in \mathbb{R}$

$\Rightarrow L[\lambda p + q](x)$

$= [(\lambda p + q)(0)](x^3) + [\lambda p + q](1)(x-4)^2$

$= [\lambda p(0) + q(0)] x^3 + [\lambda p(1) + q(1)](x-4)^2$

$= \lambda p(0)x^3 + q(0)x^3$

$+ \lambda p(1)(x-4)^2 + q(1)(x-4)^2$

$$= \lambda [P(0)x^3 + P(1)(x-4)^2]$$
$$+ [q(0)x^3 + q(1)(x-4)^2]$$

$$= \lambda L[P](x) + L[q](x)$$

b) Calcule la matriz representante de L cuando en el espacio de partida y llegada se usa la base canonica

Sol

$P \in P_3 \Rightarrow P(x) = a + bx + cx^2 + dx^3$

luego estudiemos como L afecta a la base canonica

$\{1, x, x^2, x^3\}$

$$\rightarrow L[1](x) = x^3 + (x-4)^2$$

$$= x^3 + x^2 - 8x + 16$$

$$L[x](x) = 0 \cdot x^3 + 1 \cdot (x-4)^2$$

$$= (x-4)^2$$

$$= x^2 - 8x + 16$$

$$L[x^2](x) = 0\, x^3 + 1 \cdot (x-4)^2$$

$$= x^2 - 8x + 16$$

$$L[x^3](x) = 0\, x^3 + 1 \cdot (x-4)^2$$

$$= x^2 - 8x + 16$$

$$\begin{pmatrix} 16 & 16 & 16 & 16 \\ 1 & 1 & 1 & 1 \\ -8 & -8 & -8 & -8 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

P8)

Sea $T: P_2 \to P_2$ definida por

$$T(a_0 + a_1 x + a_2 x^2)$$

$$= (a_0 + 2a_2) + a_1 x + (2a_0 + a_2)x^2$$

a) demuestre que T es lineal

Sol

$$T(\lambda[a_0 + a_1 x + a_2 x^2] + [b_0 + b_1 x + b_2 x^2])$$

$$= T([\lambda a_0 + b_0] + [\lambda a_1 + b_1]x + [\lambda a_2 + b_2]x^2)$$

$$= [\lambda a_0 + b_0 + 2[\lambda a_2 + b_2]] + (\lambda a_1 + b_1)x + (2[\lambda a_0 + b_0] + [\lambda a_2 + b_2])x^2$$

$$= \lambda a_0 + 2\lambda a_2 + \lambda a_1 x$$
$$+ 2\lambda a_0 x^2 + \lambda a_2 x^2$$

$$+ b_0 + 2b_2 + b_1 x$$
$$+ 2b_0 x^2 + b_2 x^2$$

$$= \lambda [a_0 + 2a_2 + a_1 x + 2a_0 x^2 a_2 x^2]$$
$$+ [b_0 + 2b_2 + b_1 x + 2b_0 x^2 + b_2 x^2]$$

$$= \lambda \quad T(a_0 + a_1 x + a_2 x^2)$$
$$+ T(b_0 + b_1 x + b_2 x^2) \quad //$$

b] Encuentre su matriz representente

sol

$T(1) = 1 + 2x^2$

$T(x) = x$

$T(x^2) = 2 + x^2$

$$\rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \begin{matrix} \rightarrow 1 \\ \rightarrow x \\ \rightarrow x^2 \end{matrix}$$

P9) sea $T: \mathbb{R}^4 \to \mathbb{R}^3$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ x_1 + x_3 - 3x_4 \\ x_1 + x_2 + x_3 \end{pmatrix}$$

a) Demuestre que T es lineal

Sol

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & -3 \\ 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$\therefore$ T es lineal //

b) Calcule las bases del

$Ker(T)$ e $Im(T)$

sol)

$Ker(T)$ estudiemos

$Ax = 0$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & -3 \\ 1 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & -3 \\ 0 & -3 & 1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & -1 & -2 & 6 \\ 0 & 0 & 4 & -18 \end{bmatrix}$$

$$\Rightarrow 4x_3 - 18x_4 = 0$$

$$\Rightarrow x_3 = \frac{9}{2} x_4$$

$$\Rightarrow -x_2 - 2x_3 + 6x_4 = 0$$

$$\Rightarrow x_2 = -9x_4 + 6x_4 = -3x_4$$

$$2x_1 - x_2 = 0$$

$$\Rightarrow x_1 = -\frac{3}{2} x_4$$

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -3/2 \\ -3 \\ 9/2 \\ 1 \end{pmatrix}$$

$$= \frac{x_4}{2} \begin{pmatrix} -3 \\ -6 \\ 9 \\ 2 \end{pmatrix}$$

$$\Rightarrow \ker(f) = \left\langle \left\{ \begin{pmatrix} -3 \\ -6 \\ 9 \\ 2 \end{pmatrix} \right\} \right\rangle$$

base es $\begin{pmatrix} -3 \\ -6 \\ 9 \\ 2 \end{pmatrix}$ y

$\dim(\ker(f)) = 1$

$\Rightarrow$ Por TNI $\dim(\operatorname{Im} T) = 3$

Por la parte anterior escalonamos A y deducimos que

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$ son

li $\Rightarrow$ como son 3 son base

c) Es T inyectiva, sobreyectiva

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Sol

$\dim(\ker(T)) = 1 \neq 0$

$\Rightarrow$ T no es inyectiva

$\dim(\operatorname{Im}(T)) = 3 = \dim(\mathbb{R}^3)$

$\Rightarrow$ T es sobreyectiva

d) Calcule la matriz representante con respecto a las bases de $\mathbb{R}^4$ y $\mathbb{R}^3$

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -3 \\ 1 \end{pmatrix} \right\}$$

y

$$B' = \left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

respectivamente

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sol

1- calculamos la imagen de B

$$T\begin{pmatrix}1\\0\\0\\2\end{pmatrix} = \begin{pmatrix}2 & -1 & 0 & 0\\ 1 & 0 & 1 & -3\\ 1 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\2\end{pmatrix}$$

$$= \begin{pmatrix}2\\-5\\1\end{pmatrix}$$

$$T\begin{pmatrix}2\\-1\\-3\\4\end{pmatrix} = \begin{pmatrix}2 & -1 & 0 & 0\\ 1 & 0 & 1 & -3\\ 1 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}2\\-1\\-3\\4\end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -13 \\ -2 \end{pmatrix}$$

$$f\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & -3 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$$

$$f\begin{pmatrix} 1 \\ -1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & -3 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -5 \\ -3 \end{pmatrix}$$

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ahora buscamos

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$2 = \alpha - \gamma$$

$$-5 = 2\alpha + \beta - \gamma$$

$$1 = 2\alpha - \gamma = \alpha + \alpha - \gamma = \alpha + 2$$

$$\rightarrow \alpha = -1$$

$$\rightarrow \gamma = -3$$

$$\rightarrow \beta = -5 - 2\alpha + \gamma$$

$$= -5 + 2 - 3 = -6$$

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + -6 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ -13 \\ -2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$\rightarrow \quad 5 = \alpha - \gamma$

$-13 = 2\alpha + \beta - \gamma$

$-2 = 2\alpha - \gamma = \alpha + \alpha - \gamma = \alpha + 5$

$\rightarrow \quad \alpha = -7$

$\rightarrow \quad \gamma = -12$

$\beta = -13 - 2\alpha + \gamma = -13 + 14 - 12$

$= -11$

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$$\begin{pmatrix} 5 \\ -13 \\ -2 \end{pmatrix} = -7\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 11\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 12\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$2 = \alpha - \gamma$$

$$-4 = 2\alpha + \beta - \gamma$$

$$\to 2 = 2\alpha - \gamma = \alpha + 2$$

$$\to \alpha = 0$$

$$\to \gamma = -2$$

$$\to \beta = -6$$

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$$\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 6\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -5 \\ -3 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$3 = \alpha - \gamma$$

$$-5 = 2\alpha + \beta - \gamma$$

$$-3 = 2\alpha - \gamma = \alpha + 3$$

$$\rightarrow \alpha = -6$$

$$\gamma = -9$$

$$\beta = -5 - 2\alpha + \gamma = -5 + 12 - 9 = -2$$

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$$\begin{pmatrix} 3 \\ -5 \\ -3 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 9\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 6\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ -13 \\ -2 \end{pmatrix} = -7\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 11\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 12\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 6\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -5 \\ -3 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 9\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$M \to \begin{pmatrix} -1 & -7 & 0 & -6 \\ -6 & -11 & -6 & -2 \\ -3 & -12 & -2 & -9 \end{pmatrix}$$

P10)

$T: \mathbb{R}^n \to \mathbb{R}^n$ transforma-
ción lineal

a) verifique que
$Im(T^2) \subseteq Im(T)$ y $Ker(T) \subseteq Ker(T^2)$ $[T^2 = T \circ T]$

Sol) Sea $y \in Im(T^2)$

$\Rightarrow \exists\, x$ s.t. $y = T^2(x)$

$\Rightarrow y = T(T(x))$

$\Rightarrow y = T(\bar{x}),\ \bar{x} = T(x)$

$\Rightarrow y \in \operatorname{Im}(T)$

$\therefore\quad \operatorname{Im}(T^2) \subseteq \operatorname{Im}(T)$ //

$\operatorname{Ker}(T) \subseteq \operatorname{Ker}(T^2)$

$x \in \operatorname{Ker}(T) \Rightarrow T(x) = 0$

$\Rightarrow T(T(x)) = T(0) = 0$

$\Rightarrow T^2(x) = 0 \Rightarrow x \in \operatorname{Ker}(T^2)$

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$\Rightarrow \ Ker(T) \subseteq Ker(T^2)$

b) pdq $Im(T^2) = Im(T)$

$\Rightarrow \ Ker(T^2) = Ker(T)$

Sol)

$T : \mathbb{R}^n \to \mathbb{R}^n$

por TNI

$\Rightarrow \dim(\mathbb{R}^n) = \dim(Im(T)) + \dim(Ker(T))$

$\Rightarrow n = \dim(Im(T)) + \dim(Ker(T))$

$T^2 : \mathbb{R}^n \to \mathbb{R}^n$

por TNI

$\Rightarrow n = \dim(Im(T^2)) + \dim(Ker(T^2))$

$Im(T) = Im(T^2)$

$\Rightarrow dim(Im(T)) = dim(Im(T^2))$

$\Rightarrow n = dim(Im(T)) + dim(Ker(T))$

$n = dim(Im(T^2)) + dim(Ker(T^2))$

$\Rightarrow dim(Ker(T)) = dim(Ker(T^2))$

Por *

$Ker(T) \subseteq Ker(T^2)$

y ambos son SeV

$\Rightarrow Ker(T) = Ker(T^2)$