Physics Letters B 789 (2019) 256–261

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Exact scattering waves off nonlocal potentials under Coulomb interaction within Schrödinger's integro-differential equation

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ARTICLE INFO

Article history:
Received 27 November 2018
Received in revised form 1 December 2018
Accepted 3 December 2018
Available online 21 December 2018
Editor: W. Haxton

Keywords:
Nonlocal potential
Scattering theory
Optical model potential
Integro-differential equation
Coulomb interaction

ABSTRACT

An exact solution for the scattering wavefunction from a nonlocal potential in the presence of Coulomb interaction is presented. The approach is based on the construction of a Coulomb Green's function in coordinate space whose associated kernel involves any nonlocal optical potential superposed to the Coulomb-screened interaction. The scattering wavefunction, exact solution of the integro-differential Schrödinger's equation, poses no restrictions on the type of nonlocality of the interaction nor on the beam energy.

1. Introduction

Beyond its intrinsic merit, the value of counting on an exact solution to any given problem is that it provides with accuracy benchmarks for alternative approaches. In the particular case of the interaction of a single nucleon with a nucleus it is well established that the coupling is nonlocal, feature that arises from the fermionic nature of all interacting nucleons. In the presence of a nonlocal potential Schrödinger's equation for scattering waves becomes integro-differential. Explicit treatments of nonlocalities in Schrödinger's equation is an issue that has captured increasing interest from the stand point of *ab-initio* theories and models, specially aiming to global approaches for structure and reactions [1–3]. Therefore, robust methods able to provide solutions to the wave equation for any kind of kernel become imperative to accurately treat and assess model-independent nonlocalities of nuclear interactions. To this date, however, it can safely be stated that the only established kernel-independent approach that solves exactly Schrödinger's equation for the wavefunction – in the presence of Coulomb interaction – is the one reported in Refs. [4,5]. The method is based on finite difference techniques, reducing the problem to a matrix equation for the wavefunction. In this work we present an alternative solution to the integro-differential equation, resulting in a non-singular integral equation readily invertible. The key feature in this case is the construction of a Green's function capable of accounting exactly for the underlying long-range Coulomb interaction.

In the context of nucleon–nucleus scattering, physical quantities of major interest are the scattering amplitudes and wavefunctions. The latter being useful, for example, in distorted wave Born approximation applications. When expressed in coordinate space the equation for the wavefunction becomes integro-differential. Early solutions to this problem were proposed by Perey and Buck [6], transforming the non-local potential by a local-equivalent. A shortcoming of this approach is that the calculated outgoing wavefunction differs from the exact one, distortion which is known as Perey effect and characterized by the Perey correction factor [7].

Other solutions to Schrödinger's integro-differential equation follow iterative procedures [7,8]. In these schemes Schrödinger's differential equation is integrated with a non-homogeneous term consisting of the projection of the nonlocal potential onto an intermediate solution, $U_{nl}|\chi_i\rangle$. These procedures begin with a given seed to generate the starting solution $|\chi_0\rangle$, with subsequent iterations until convergence is reached. These iterative methods may require prior knowledge of the solution in order to make convergence more efficient, though there is no guarantee to converge to the correct solution. In the case of Ref. [9], a mean-value technique is applied to approximate $U_{nl}|\chi_i\rangle$, reducing the problem to a second-order homogeneous differential equation. Quite

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https://doi.org/10.1016/j.physletb.2018.12.004

recently another approach has been proposed to deal with nonlocal potentials [10], where a Taylor approximation for the radial wave function is applied. This strategy is based on the assumption that nonlocality is dominant around the diagonal in coordinate space, a non universal feature as reported in Ref. [11] for microscopic potentials based on off-shell g matrices.

Another method to calculate waves off nonlocal potentials in the presence of long-range Coulomb interaction is that of Refs. [12, 13], where Lanczos technique is used to solve integral equations derived from the nonlocal Schrödinger equation. More recently, in Refs. [14,15] a numerical treatment to this problem has been proposed with the use of Berggren basis, where an off-diagonal approximation is used to control the Coulomb singularity along the diagonal in momentum space. Applications of this approach have been restricted to low energies and intermediate mass targets.

Solutions to the scattering problem in momentum space have also been investigated [16–22]. See Ref. [23] for a review on the subject. While an advantage of momentum-space approaches is that nonlocalities are naturally accounted for, one of its drawbacks is that no method is available to extract the associated scattering waves. In the absence of Coulomb interaction the calculation of scattering amplitudes is rather straightforward, reducing the problem to a Lippmann–Schwinger integral equation for the scattering matrix. However, in the presence of Coulomb potential the approach cannot be applied right away due to the $\sim 1/q^2$ singularity of the interaction. An exact solution addressing this singularity has been proposed by Vincent and Phatak by means of a cut-off technique to the Coulomb long-range tail [24]. This approach has been applied to proton–nucleus scattering at intermediate energies [17], where its accuracy is significantly improved after a detailed multipole treatment of the charge form factor convoluted with a sharp cut-off potential [25].

In this article we present exact solutions for scattering waves off any finite-range nonlocal potential in coordinate space, where the Coulomb interaction is included without approximation. The approach, briefly sketched in an appendix of Ref. [26] in the context of quasielastic (p,n) charge-exchange reactions, is not restricted on energy of the projectile, charge of the colliding particles nor nature of the nonlocality.

This paper is organized as follows. In Sec. 2 we lay out the framework and present a formal solution to the scattering problem with nonlocal potentials in the presence of Coulomb interactions. We provide a demonstration of the solution and illustrate its consistency with a numerical example. In Sec. 3 we present the main conclusions of the work.

2. Integral equation for scattering waves

Let us consider the collision of a proton with a nucleus of charge Ze. The interaction U between them is given by the sum of a pure hadronic contribution (U_H) and the Coulomb interaction (U_C) due to the charge distribution of the nucleus, $U = U_H + U_C$. The hadronic part is regarded in general as a nonlocal operator so that the total potential can be cast as the sum of a point-Coulomb and short-range terms,

$$U(\mathbf{r}',\mathbf{r}) = U^{[s]}(\mathbf{r}',\mathbf{r}) + \frac{\beta}{r}\delta(\mathbf{r}'-\mathbf{r}) \,, \tag{1}$$

with $\beta = Ze^2$. Here $U^{[s]}$ defines the finite-range part of the interaction where the point-Coulomb interaction has been subtracted, namely $U^{[s]} = U_H + U_C - \beta\delta(\mathbf{r}'-\mathbf{r})/r$.

With the above construction in mind we examine Schrödinger's equation for scattering waves, which in coordinate representation reads

$$-\nabla^2\psi_{\mathbf{k}}(\mathbf{r}) + \frac{2m}{\hbar^2}\int d\mathbf{r}' U(\mathbf{r},\mathbf{r}')\psi_{\mathbf{k}}(\mathbf{r}') = k^2\psi_{\mathbf{k}}(\mathbf{r}) \,, \tag{2}$$

with m the nucleon–nucleus reduced mass and k the asymptotic relative momentum. Spin and isospin variables are omitted for notation simplicity. Considering a spin-0 closed-shell target interacting with a spin-$\frac{1}{2}$ nucleon, the following partial wave expansion for the scattering wavefunction becomes suitable,

$$\psi_{\mathbf{k}}(\mathbf{r}) = \sqrt{\frac{2}{\pi}}\sum_{jlm} i^l \mathcal{Y}^m_{jl1/2}(\hat{\mathbf{r}})e^{i\sigma_l}\frac{u_{jl}(r)}{r}\mathcal{Y}^{m\dagger}_{jl1/2}(\hat{\mathbf{k}}) \,. \tag{3}$$

In this expansion $\mathcal{Y}^m_{jl1/2}$ denotes spherical vectors and σ_l the Coulomb phase-shift for partial wave l. Here $u_{jl}(r)$ is the radial wavefunction. In the limit where the finite-range interaction $U^{[s]}$ is set to zero, the unperturbed wavefunction becomes a free Coulomb wave due to a pointlike source, $\psi_{\mathbf{k}}(\mathbf{r}) \to \phi_c(\mathbf{r})$, where

$$\phi_c(\mathbf{r}) = \sqrt{\frac{2}{\pi}}\sum_{jlm} i^l \mathcal{Y}^m_{jl1/2}(\hat{\mathbf{r}})e^{i\sigma_l}F_l(kr)\mathcal{Y}^{m\dagger}_{jl1/2}(\hat{\mathbf{k}}) \,, \tag{4}$$

with F_l the regular Coulomb function. In the absence of Coulomb interaction ($\beta = 0$), this expression leads to normalized plane waves $\phi_{\mathbf{k}}(\mathbf{r})$,

$$\phi_{\mathbf{k}}(\mathbf{r}) = \frac{\mathbf{1}_\sigma}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{r}} \,, \tag{5}$$

with $\mathbf{1}_\sigma$ the identity in spin-$\frac{1}{2}$-space.

By replacing $\psi_{\mathbf{k}}(\mathbf{r})$ from Eq. (3) into Eq. (2), following standard procedures we obtain [27]

$$\left[\frac{1}{r}\left(\frac{d^2}{dr^2}\right)r - \frac{l(l+1)}{r^2} + k^2\right]\frac{u_{jl}(r)}{r} = \frac{2m}{\hbar^2}\int_0^\infty r'\,dr' U_{jl}(r,r')u_{jl}(r') \,, \tag{6}$$

where the multipoles U_{jl} of the interaction are obtained from

$$U_{jl}(r',r) = \iint d\hat{\mathbf{r}}\,d\hat{\mathbf{r}}'\mathcal{Y}^{m\dagger}_{jl1/2}(\hat{\mathbf{r}}')U(\mathbf{r}',\mathbf{r})\mathcal{Y}^m_{jl1/2}(\hat{\mathbf{r}}) \,. \tag{7}$$

Making explicit the separation of the interaction into a pointlike source and finite-range remaining

$$U_{jl}(r',r) \equiv U^{[s]}_{jl}(r',r) + \frac{\beta}{r^3}\delta(r'-r) \,, \tag{8}$$

we obtain

$$\begin{aligned}\mathcal{D}_c u_{jl}(r) &\equiv \left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - \frac{2k\eta}{r} + k^2\right]u_{jl}(r) \\ &= \frac{2m}{\hbar^2}\int dr' r U^{[s]}_{jl}(r,r')r'u_{jl}(r') \,.\end{aligned} \tag{9}$$

Here $\mathcal{D}_c$ denotes a second order differential operator which includes the point-Coulomb contribution, with the Sommerfeld parameter η given by $\eta = m\beta/\hbar^2 k$. Two linearly independent homogeneous solutions to Eq. (9) are the regular (F_l) and irregular (G_l) Coulomb wavefunctions which satisfy $\mathcal{D}_c F_l(kr) = \mathcal{D}_c G_l(kr) = 0$. We adopt phase conventions such that their asymptotic behavior are given by

$$\begin{aligned}F_l(z)\big|_{z\to\infty} &\to \sin(z - l\pi/2 - \eta\ln 2z + \sigma_l) \,, \\ G_l(z)\big|_{z\to\infty} &\to \cos(z - l\pi/2 - \eta\ln 2z + \sigma_l) \,.\end{aligned} \tag{10}$$

2.1. Formal solution

We now look for a solution for the scattering wavefunctions in the presence of the Coulomb term. Let us first recall the case where the Coulomb interaction is suppressed. In such a case, if $\hat{V}$ represents a short-range potential, the Lippmann–Schwinger integral equation for scattering waves $|\psi\rangle$ at a given energy E reads

$$|\psi\rangle = |\phi_0\rangle + \hat{G}_0(E+i\eta)\hat{V}|\psi\rangle\,, \tag{11}$$

where $|\phi_0\rangle$ represents free incoming waves and $\hat{G}_0(E+i\eta) = (E+i\eta-\hat{K})^{-1}$, corresponding to the free propagator. Here $\hat{K}$ is the kinetic energy operator, so that $\hat{K}|\boldsymbol{k}\rangle = (k^2/2m)|\boldsymbol{k}\rangle$. To obtain the scattering waves in coordinate space it is customary to evaluate the free propagator in coordinate representation, i.e. $\langle\boldsymbol{r}|\hat{G}_0(E+i\eta)|\boldsymbol{r}'\rangle$. Following Joachain [27], after performing partial wave expansions and subsequent contour integrations in the complex k-plane it is found that

$$\langle\boldsymbol{r}|\hat{G}_0(E+i\eta)|\boldsymbol{r}'\rangle = \frac{2m}{\hbar^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(-\frac{i}{k}\right) j_l(kr_<)h_l^{(+)}(kr_>)Y_{lm}(\hat{\boldsymbol{r}})Y_{lm}^*(\hat{\boldsymbol{r}}')\,, \tag{12}$$

where $h_l^{(+)} = j_l - in_l$. Here j_l and n_l denote spherical Bessel and Neumann functions, respectively. Additionally, $r_< = \min\{r, r'\}$, while $r_> = \max\{r, r'\}$. If we now include a Coulomb interaction, then Eq. (11) for the wavefunction can be cast as

$$|\psi\rangle = |\chi_0\rangle + \hat{G}_C(E+i\eta)\hat{U}^{[s]}|\psi\rangle\,, \tag{13}$$

where $|\chi_0\rangle$ correspond to free incoming Coulomb waves, and

$$\hat{G}_C(E+i\eta) = \frac{1}{E+i\eta-\hat{K}-\hat{V}_c}\,, \tag{14}$$

to the free Coulomb propagator. In this case, $\hat{V}_C$ corresponds to the point Coulomb interaction and $U^{[s]}$ is defined in Eq. (1). The difficulty in this case is that there is no known procedure to obtain $\langle\boldsymbol{r}'|\hat{G}_C(E+i\eta)|\boldsymbol{r}\rangle$, in analogy to the one adopted to obtain Eq. (12) for the propagator. Most of the difficulty arises from the fact that $\hat{K}$ does not commute with $\hat{V}_C$, preventing manageable contour integrations in the complex k-plane.

To circumvent the above difficulty with Coulomb interactions, we look for a solution for outgoing scattering waves u_{jl} in Eq. (9), expressed as the superposition of homogeneous and particular solutions in the form

$$u_{jl}(r) = \frac{1}{k}F_l(kr) + \frac{2m}{\hbar^2}\iint dr'dr''G_l^{c(+)}(r,r';k)\left[r'U_{jl}^{[s]}(r',r'')r''\right]u_{jl}(r'')\,. \tag{15}$$

For the construction of a particular solution we pursue the following ansatz for the Coulomb propagator $G_l^{c(+)}$ in partial wave l,

$$G_l^{c(+)}(r,r';k) = -\frac{i}{k}F_l(kr_<)\mathcal{H}_l^{(+)}(kr_>)\,, \tag{16}$$

where $\mathcal{H}_l^{(+)} = F_l - iG_l$.

The validity of this ansatz for $G_l^{c(+)}$ calls for a demonstration. To do so, we verify that the formal solution expressed by Eq. (15) for u_{jl}, satisfies the integro-differential equation in Eq. (9). Hence, let us examine the action of $\mathcal{D}_c$ on F_l and the integral involving the kernel. Since F_l satisfies $\mathcal{D}_cF_l = 0$, then we just need to focus on

$$\mathcal{Z}(r) \equiv \mathcal{D}_c\int_0^{\infty} dr'G_l^{c(+)}(r,r';k)W_{jl}(r')\,, \tag{17}$$

where $W_{jl}(r')$ represents the integral over r'' given by

$$W_{jl}(r') \equiv \frac{2m}{\hbar^2}\int_0^{\infty} r'U_{jl}^{[s]}(r',r'')r''u_{jl}(r'')dr''\,. \tag{18}$$

Making explicit $G_l^{c(+)}$ defined in Eq. (16) by splitting the integral over r' in Eq. (17) into two sub-intervals, $[0,r]$ and $[r,\infty)$, we get

$$\int_0^{\infty} G_l^{c(+)}(r,r';k)W_{jl}(r')dr'dr'' = -\frac{i}{k}\left[\mathcal{H}_l^{(+)}(kr)\int_0^{r} dr'F_l(kr')W_{jl}(r') + F_l(kr)\int_r^{\infty} dr'\mathcal{H}_l^{(+)}(kr')W_{jl}(r')\right]. \tag{19}$$

Taking derivatives with respect to r and using the Wronskian identity

$$F_l(z)\mathcal{H}_l^{(+)\prime}(z) - F_l'(z)\mathcal{H}_l^{(+)}(z) = i\,, \tag{20}$$

we obtain

$$-\frac{\partial^2}{\partial r^2}\int_0^{\infty} G_l^{c(+)}(r,r';k)W_{jl}(r')dr' = W_{jl}(r) + \frac{i}{k}\left[\frac{\partial^2\mathcal{H}_l^{(+)}(kr)}{\partial r^2}\int_0^{r} dr'F_l(kr')W_{jl}(r') + \frac{\partial^2F_l(kr)}{\partial r^2}\int_r^{\infty} dr'\mathcal{H}_l^{(+)}(kr')W_{jl}(r')\right]. \tag{21}$$

Combining this result with Eq. (19) and considering that $\mathcal{D}_cF_l = \mathcal{D}_c\mathcal{H}_l^{(\pm)} = 0$, we get

$$\mathcal{Z}(r) = \mathcal{D}_c\int G_l^{c(+)}(r,r';k)W_{jl}(r')dr' = W_{jl}(r)\,, \tag{22}$$

proving that u_{jl} as given by Eq. (15) constitutes the solution to the wave equation (9) for outgoing scattering waves.

An appealing feature of the propagator expressed by Eq. (16) is that it is non-singular, being a continuous function of r and r'. The gradient of $G_l^{c(+)}$ is discontinuous at the diagonal $r = r'$, although this feature poses no particular drawback. Note that Eq. (15) takes the form of a Lippmann–Schwinger integral equation for scattering waves in the presence of Coulomb interaction, which we recast as

$$\int dr''\left[\delta(r-r'') - K_{jl}(r,r'')\right]u_{jl}(r'') = \tfrac{1}{k}F_l(kr)\,, \tag{23}$$

where the kernel K_{jl} is given by

$$K_{jl}(r,r'') = \frac{2m}{\hbar^2}\int dr'G_l^{c(+)}(r,r';k)\left[r'U_{jl}^{[s]}(r',r'')r''\right]. \tag{24}$$

This kernel contains the nonlocal hadronic interaction superposed to the Coulomb-screened electrostatic interaction. Note that Eq. (23) enables to obtain the actual scattering wavefunction, solution of Schrödinger's integro-differential equation, by means of direct matrix inversion. In this context, the solutions for the scattering waves are exact. The novel feature here is that Coulomb interaction is also treated exactly.

The solution for u_{jl} from Eq. (23) enables the calculation of the scattering amplitude, which follows from the asymptotic form of Eq. (15), where r is taken far away from the scattering center. In this limit we have

$$G_l^{c(+)}(r,r';k)\big|_{r\gg r'} \longrightarrow -\frac{i}{k}F_l(kr')\mathcal{H}_l^{(+)}(kr)\,, \tag{25}$$

which once replaced in Eq. (15) for u_{jl} yields

$$k\,u_{jl}(r)\big|_{r\to\infty} \to F_l(kr) + \Delta_{jl}\left[F_l(kr) \mp iG_l(kr)\right]\,, \tag{26}$$

with

$$\Delta_{jl} = -\frac{2mi}{\hbar^2}\iint r'dr'\,r''dr''F_l(kr')U_{jl}^{[s]}(r',r'')u_{jl}(r'')\,. \tag{27}$$

These last two relations allow independent ways to obtain Δ_{jl}. The latter involves direct integration of the wavefunction whereas the former evaluates asymptotically the ratio

$$\Delta_{jl} = \frac{ku_{jl}(r) - F_l(kr)}{F_l(kr) - iG_l(kr)}\,, \tag{28}$$

for sufficiently large r. These equivalent forms to calculate Δ_{jl} serve as a means to crosscheck consistency of the solutions. Once Δ_{jl} is obtained, the scattering amplitude f_{jl} and short-range phase shift $\hat{\delta}_{jl}$ follow from

$$\Delta_{jl} = ikf_{jl} = \tfrac{1}{2}\left(e^{2i\hat{\delta}_{jl}} - 1\right)\,. \tag{29}$$

2.2. Numerical application

To illustrate the consistency of the solution expressed by Eq. (15) under nonlocal interactions, we present applications for p+^{40}Ca elastic scattering at 30.3 and 300 MeV beam energies. For these examples we choose microscopic optical model potentials taken from momentum-space *in-medium* folding calculations, where the mixed density of the target is folded to the full off-shell g matrix, accounting for the Fermi motion of target nucleons [28]. The bare nucleon–nucleon interaction used to calculate fully off-shell g matrices is Argonne v_{18} [29]. The optical potential is then transformed to coordinate space as described in Ref. [11], resulting in nonlocal potentials with intricate structure, depending on the momentum cutoff used in the Fourier transform. The Coulomb interaction corresponds to that due to a uniform charge distribution. No localization of hadronic contributions is performed at any stage of the calculations.

The numerical implementation of Eq. (23) follows from the discretization of r and r' over an N-point uniform mesh, where $r \to r_n = nh$, with h a suitable spacing. Trapezoidal rule is adequate in this case. The kernel, function of r and r', becomes a finite $N\times N$ matrix which we denote as $\mathbb{K}$. The solution to Eq. (23) takes the form

$$\mathbf{u} = (1-\mathbb{K})^{-1}\mathbf{u}_0\,, \tag{30}$$

with $\mathbf{u}_0$ the unperturbed wave $F_l(kr)/k$, while $\mathbf{u}$ denotes the scattering wave over the discrete mesh. In this case we use $N = 150$, with spacing $h = 0.1$ fm. Note that the scattering wavefunction is fully determined from Eq. (30), requiring no normalization to match asymptotic waves. Results from this approach (referred in the following as Exact Scattering Waves, ESW) are compared with those obtained from DWBA98 code [4], which provides exact numerical solutions for Schrödinger's integro-differential nonlocal wave equation.

In Fig. 1 we show results for the ratio-to-Rutherford of the elastic cross section $\sigma(\theta)/\sigma_R(\theta)$ (a), (b) and analyzing power A_y (c), (d) as functions of the scattering angle in the center-of-mass reference frame. Frames on the left-hand side correspond to 30.3 MeV proton scattering off ^{40}Ca, and those on the right-hand side correspond to 300 MeV. Solid curves represent results based on the present approach (ESW), while dashed curves represent solutions using DWBA98 code [4]. In the case of 30.3 MeV we observe that differences in σ/σ_R become slightly noticed for $\theta_{c.m.} > 140°$. In the case of the analyzing power, differences are quite moderate but enough to distinguish the two approaches. Results for proton scattering at 300 MeV are plotted up to $\theta_{c.m.} = 60°$, corresponding to a relatively high momentum transfer of 4 fm^{-1}. In this case we note that the curves for both $\sigma(\theta)/\sigma_R(\theta)$ and A_y overlap almost completely, illustrating the level of agreement for the two exact approaches.

In the context of the numerical application at 30.3 MeV, we have also investigated the use of NLAT code [8], developed to solve the nonlocal Schrödinger equation using an iterative procedure. Results from this code using Perey–Buck-type potential in the version developed by Tian *et al.* [30] are in reasonable agreement with the ones obtained with ESW and DWBA98 approaches. This is illustrated in inset (e) of Fig. 1, where we plot the ratio-to-Rutherford of the elastic cross section. Black, red and blue curves denote results for NLAT, DWBA98 and ESW, respectively, displaying reasonable agreement among them. However, when NLAT code is used for the microscopic model it fails to solve the nonlocal equation. Inset (e) shows results from two trial local potentials proposed in the regular input of NLAT. One is Koning–Delaroche (KD) potential [31] (dashed curve) and the other Chapel–Hill potential (CH89) [32] (dotted curve). As observed, these trial solutions lead to different solutions for the cross sections, demonstrating the sensitivity to the kernel-shape of NLAT approach in its present version. It is worth noting that the cross section obtained from ESW using the microscopic potential is very similar to that from Perey–Buck–Tian nonlocal potential, so one would expect KD to be a reasonable trial potential in the microscopic case as well.

3. Discussion and concluding remarks

The solution embodied by Eq. (15) for the scattering waves in the presence of Coulomb interactions is a piece of knowledge overlooked in the field. As demonstrated, this equation leads univocally to *the solution* for the scattering waves. By contrast, any iterative method can always be re-expressed as an infinite series, being also equivalent to a perturbative approach. Assessing beforehand its convergence is an issue with no formal solution. In order to anticipate the convergence of any iterative method one needs information on the initial guess in addition to the structure of the kernel. At the end, their effectiveness relies on empirical know-how under controlled scenarios.

In summary, we have presented an exact solution for the scattering waves off nonlocal optical potentials in the presence of long-range Coulomb interaction. The structure of the solution poses no restrictions on the type of nonlocality, beam energy nor charge of colliding particles. Its numerical implementation leads to non-singular finite matrices over a spatial mesh, allowing to obtain the scattering waves by direct matrix inversion. When compared to exact solutions of the integro-differential Schrödinger's equation provided by the DWBA98 code, excellent agreement is observed

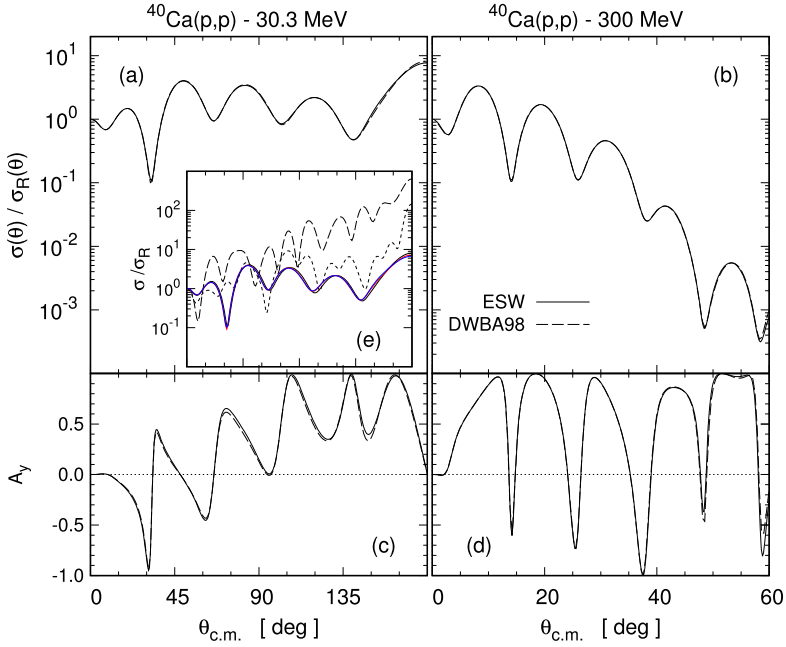

Fig. 1. Calculated ratio-to-Rutherford elastic cross section (a), (b) and analyzing power (c), (d) as functions of the center-of-mass scattering angle. Results obtained from microscopic nonlocal potential for ^{40}Ca(p, p) elastic scattering at 30.3 and 300 MeV. Solid and dashed curves denote results from ESW (this work) and DWBA98, respectively. Inset (e) shows results for σ/σ_R for Perey–Buck-type potential obtained from ESW (solid blue curves), DWBA98 (solid red curve) and NLAT (solid black curves). Inset (e) also shows results for microscopic optical model obtained with NLAT using KD [31] and CH89 [32] potentials as starting solutions in the iterative procedure, denoted with dashed and dotted curves, respectively.

in the calculated scattering observables at nucleon energies of up to 300 MeV. With these features, the solution we present provides benchmark solutions to compare with. Since the approach we present leads to actual solutions for the scattering waves, it is well suited for distorted-wave Born approximation for nuclear reactions. Additionally, the approach presented here is well suited for coupled-channels [26], with extension to inelastic processes underway [33].

Acknowledgement

H.F.A. thanks colleagues at CEA, Bruyères-le-Châtel, France, for their hospitality during his stay where part of this work was done.

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