* Pauta Auxiliar 7

"Cálculo Avanzado y Aplicaciones"

* Repaso Números complejos:

- Forma cartesiana: $z = x + iy$; $\operatorname{Re}(z) = x$, $\operatorname{Im}(z) = y$.
- Forma polar: $z = |z| \cdot e^{i\arg(z)}$; $|z| = \sqrt{x^2+y^2}$, $\arg(z) = \arctan\left(\frac{y}{x}\right) = \theta$
- Conjugado: $\bar{z} = x - iy = |z| \cdot e^{-i\theta}$
- Propiedades:

$$|z|^2 = z\bar{z} \quad ; \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad ; \quad |z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

- Suma:

$$(a+bi) \pm (c+di) = (a \pm c) + i(b \pm d)$$

- Multiplicación:

$$(a+bi) \cdot (c+di) = (ac-bd) + i(bc+ad)$$

- División:

$$\frac{(a+bi)}{(c+di)} = \frac{(a+bi)\cdot(c-di)}{c^2+d^2} \Rightarrow \frac{(a+bi)}{(c+di)} = \frac{ac+bd}{c^2+d^2} + i\left(\frac{bc-ad}{c^2+d^2}\right)$$

- De Moivre:

$$z^n = |z|^n e^{in\theta} = |z|^n(\cos(n\theta) + i\sin(n\theta)) = |z|^n(\cos(\theta) + i\sin(\theta))^n$$

- Raíces de la unidad:

$$z^n = 1 \Rightarrow z = e^{\frac{2\pi i k}{n}} \; ; \; k = 0, 1, \ldots, n-1 \quad / \quad \sum_{k=0}^{n-1} e^{\frac{2\pi i k}{n}} = 0$$

P1 (a) Representando en notación cartesiana tenemos $z = x+iy$, entonces:

$$z + \frac{1}{z} = x+iy + \frac{1}{x+iy} = x+iy + \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow z + \frac{1}{z} = \left(x + \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right)$$

$$\Rightarrow \operatorname{Im}\left(z+\tfrac{1}{z}\right) = y\left(1 - \frac{1}{x^2+y^2}\right) = y\left(\frac{x^2+y^2-1}{x^2+y^2}\right)$$

Tenemos $y = \operatorname{Im}(z) > 0$ y $\sqrt{x^2+y^2} = |z| > 1$, por lo tanto, $\operatorname{Im}(z+\frac{1}{z}) > 0$ //

(b) Usando notación cartesiana:

$$|z| = |z+1| \Rightarrow \sqrt{x^2+y^2} = \sqrt{(x+1)^2+y^2} \Rightarrow x^2 = (x+1)^2$$

$$\Rightarrow x^2 - (x+1)^2 = 0 \Rightarrow (x+x+1)(x-(x+1)) = 0$$

$$\Rightarrow (2x+1)\cdot(-1) = 0 \Rightarrow 2x+1 = 0 \Rightarrow x = -\frac{1}{2} \Rightarrow \boxed{\operatorname{Re}(z) = -\frac{1}{2}}$$ //

(c) Demostramos ambas implicancias:

(i) $\Rightarrow$) $|z+i| = |z-i| \Rightarrow \sqrt{x^2+(y+1)^2} = \sqrt{x^2+(y-1)^2}$

$$\Rightarrow (y+1)^2 = (y-1)^2 \Rightarrow (y+1)^2 - (y-1)^2 = 0 \Rightarrow (y+\cancel{1}+y-\cancel{1})(y+1-(y-1)) = 0$$

$$\Rightarrow (2y)\cdot(\cancel{y}+1-\cancel{y}+1) = 0 \Rightarrow 2y\cdot 2 = 0 \Rightarrow y = 0 \Rightarrow \boxed{z \in \mathbb{R}}$$

$\Leftarrow$) $z \in \mathbb{R} \Rightarrow z = x \Rightarrow z+i = x+i$; $z-i = x-i$

$$|z+i| = \sqrt{x^2+1} \;;\; |z-i| = \sqrt{x^2+1} \Rightarrow \boxed{|z+i| = |z-i|}$$

Por lo tanto, $\boxed{|z+i| = |z-i| \Leftrightarrow z \in \mathbb{R}}$ //.

P2 (a) $z^4 = 1$ (Raíces de la unidad)

$$z_k = e^{\frac{2\pi i k}{4}} \; ; \; k = 0,1,2,3 \Rightarrow z_0 = e^0 \Rightarrow \boxed{z_1 = 1}$$

$$z_1 = e^{\frac{2\pi i}{4}} = e^{\frac{\pi i}{2}} \Rightarrow \boxed{z = i}$$

$$z_2 = e^{\frac{4\pi i}{4}} = e^{\pi i} \Rightarrow \boxed{z_3 = -1} \; ; \; z_3 = e^{\frac{6\pi i}{4}} = e^{\frac{3\pi i}{2}} \Rightarrow \boxed{z_4 = -i}$$

(b) $(az+b)^3 = c = c \cdot 1 \Rightarrow \boxed{(az_k+b) = \sqrt[3]{c} \cdot e^{\frac{2\pi i k}{3}}}$

$$\Rightarrow az_0 + b = \sqrt[3]{c} \Rightarrow \boxed{z_0 = \frac{\sqrt[3]{c} - b}{a}}$$

$$\Rightarrow az_1 + b = \sqrt[3]{c} \cdot e^{\frac{2\pi i}{3}} \Rightarrow \boxed{z_1 = \frac{\sqrt[3]{c}\, e^{\frac{2\pi i}{3}} - b}{a}}$$

$$\Rightarrow az_2 + b = \sqrt[3]{c}\, e^{\frac{4\pi i}{3}} \Rightarrow \boxed{z_2 = \frac{\sqrt[3]{c}\, e^{\frac{4\pi i}{3}} - b}{a}}$$

(c) $z^4 + 2z^2 + 2 = 0 \; / \; w = z^2$

$$\Rightarrow w^2 + 2w + 2 = 0 \Rightarrow w = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$\Rightarrow w_1 = -1 + i \Rightarrow z_1^2 = -1 + i = \sqrt{2}\, e^{\frac{3\pi i}{4}}$$

$$w_2 = -1 - i \Rightarrow z_2^2 = -1 - i = \sqrt{2}\, e^{\frac{5\pi i}{4}}$$

$$\Rightarrow z_{1,k} = \sqrt[4]{2} \cdot e^{\frac{3\pi i}{8}} \cdot e^{\frac{2\pi i k}{2}} \; ; \; k = 0,1 \Rightarrow \boxed{z_{1,0} = \sqrt[4]{2}\, e^{\frac{3\pi i}{8}} \; ; \; z_{1,1} = \sqrt[4]{2} \cdot e^{\frac{11\pi i}{8}}}$$

$$z_{2,k} = \sqrt[4]{2}\, e^{\frac{5\pi i}{8}} e^{\frac{2\pi i k}{2}} \; ; \; k = 0,1 \Rightarrow \boxed{z_{2,0} = \sqrt[4]{2}\, e^{\frac{5\pi i}{8}} \; ; \; z_{2,1} = \sqrt[4]{2}\, e^{\frac{13\pi i}{8}}}$$

P3 (a) Resolviendo el polinomio:

$$z^2-2z+2=0 \Rightarrow z=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm\sqrt{-4}}{2}=\frac{2\pm 2i}{2}\Rightarrow z=1\pm i$$

Entonces, se tiene que:

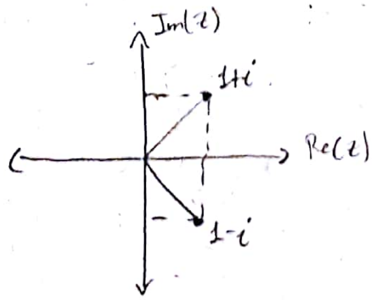
$$\frac{(\cot(\theta)+1+i-1)^n-(\cot(\theta)+1-i-1)^n}{1+i-(1-i)}=\frac{(\cot(\theta)+i)^n-(\cot(\theta)-i)^n}{2i}$$

$$=\frac{\left(\frac{\cos(\theta)}{\operatorname{sen}(\theta)}+i\right)^n-\left(\frac{\cos(\theta)}{\operatorname{sen}(\theta)}-i\right)^n}{2i}=\left(\frac{\cos(\theta)+i\operatorname{sen}(\theta)}{\operatorname{sen}(\theta)}\right)^n-\left(\frac{\cos(\theta)-i\operatorname{sen}(\theta)}{\operatorname{sen}(\theta)}\right)^n$$

$$=\frac{1}{2i\operatorname{sen}^n(\theta)}\cdot(\cos(\theta)+i\operatorname{sen}(\theta))^n-(\cos(\theta)-i\operatorname{sen}(\theta))^n=\frac{\cancel{\cos(n\theta)}+i\operatorname{sen}(n\theta)-\cancel{\cos(n\theta)}+i\operatorname{sen}(n\theta)}{2i\operatorname{sen}^n(\theta)}$$

$$=\frac{\cancel{2i}\operatorname{sen}(n\theta)}{\cancel{2i}\operatorname{sen}^n(\theta)}\Rightarrow\boxed{\frac{(\cot(\theta)+z_1-1)^n-(\cot(\theta)+z_2-1)^n}{z_1-z_2}=\operatorname{sen}(n\theta)\operatorname{cosec}^n(\theta)}$$

(b) Viendo en el gráfico, notamos que $1+i=\sqrt{2}\,e^{\frac{i\pi}{4}}$ y $1-i=\sqrt{2}\,e^{\frac{-i\pi}{4}}$. Usando De Moivre:



$$(1+i)^n+(1-i)^n=\sqrt{2}^n\left(\cos\left(\tfrac{n\pi}{4}\right)+\cancel{i\operatorname{sen}\left(\tfrac{n\pi}{4}\right)}\right)+\sqrt{2}^n\left(\cos\left(\tfrac{n\pi}{4}\right)-\cancel{i\operatorname{sen}\left(\tfrac{n\pi}{4}\right)}\right)$$

$$\Rightarrow\boxed{(1+i)^n+(1-i)^n=2\sqrt{2}^n\cos\left(\tfrac{n\pi}{4}\right)}\rightarrow$$ Puramente real...

P4 El módulo puede separarse en divisiones/multiplicaciones, entonces:

$$\left|\frac{z-2}{z+1}\right| = 2 \Rightarrow \frac{|z-2|}{|z+1|} = 2 \Rightarrow 2|z+1| = |z-2|$$

Usando representación cartesiana:

$$2|(x+1)+iy| = |(x-2)+iy| \Rightarrow 2\sqrt{(x+1)^2+y^2} = \sqrt{(x-2)^2+y^2} \quad /(\,)^2$$

$$\Rightarrow 4(x+1)^2 + 4y^2 = (x-2)^2 + y^2$$

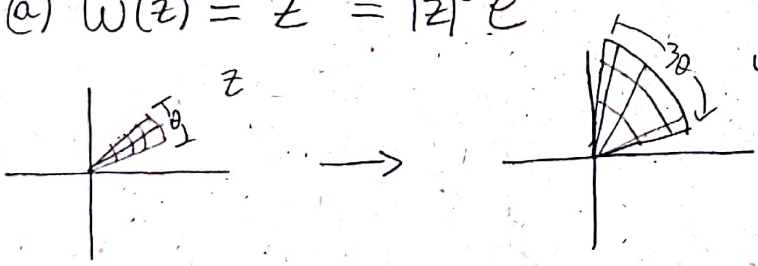
$$\Rightarrow (2x+2)^2 - (x-2)^2 = -3y^2 \Rightarrow (2x\cancel{+2}+x\cancel{-2})\cdot(2x+2-x+2) = -3y^2$$

$$\Rightarrow \cancel{3}x(x+4) = -\cancel{3}y^2 \Rightarrow x^2+4x+y^2 = 0$$

$$\Rightarrow x^2+4x+4-4+y^2=0 \Rightarrow \boxed{(x+2)^2+y^2=4}$$

↳ Circunferencia de radio $r=2$ en centro $C=(-2,0)$ //

P5 (a) $w(z) = z^3 = |z|^3 e^{i3\theta}$



(b) $w(z) = \frac{1}{z} = \frac{1}{|z|} \cdot e^{-i\theta}$