P1 Integral de una función impar sobre un dominio simétrico

Sea $A \subseteq \mathbb{R}^n$ un conjunto simétrico (es decir $x \in A$ ssi $-x \in A$) si $f: A \to \mathbb{R}$ es una función impar ($f(x) = -f(-x)\ \forall x \in A$) continua y acotada, entonces

$$\int_A f = 0$$

Sol,

Sea $h(\vec{x}) = -\vec{x} \to Jh(\vec{x}) = -\mathbb{I}$

$\Rightarrow$ por TCV

$$\int_A f(\vec{y})\,d\vec{y} = \int_{-A} f(-\vec{x})\,|\det(Jh(\vec{x}))|\,d\vec{x}$$

$$\to \int_A f(x)\,dx = \int_{-A} f(-x)\,dx$$

Ya que $|\det J_h| = |-1| = 1$

$$\Rightarrow \int_A f(x)\,dx = \int_{-A} -f(x)\,dx \to \text{Impar.}$$

$$= \int_A -f(x)\,dx \to A \text{ simetrica}$$

$$= -\int_A f(x)\,dx$$

$$\Rightarrow 2\int_A f(x)\,dx = 0$$

$$\Rightarrow \int_A f(x)\,dx = 0$$

$$\Rightarrow \int_A f = 0$$

P2

a) Sean $a, b > 0$ considere la siguiente transformación de coordenadas

$$T(\rho, \phi) = (a\rho\cos(\phi), b\rho\sin(\phi))$$

Muestre que se cumple:

$$|\det(DT(\rho, \phi))| = ab\rho$$

Sol)

$$T(\rho, \phi) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \rho\cos(\phi) \\ \rho\sin(\phi) \end{pmatrix}$$

$$\Rightarrow DT(\rho, \phi) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot D\tilde{T}(\rho, \phi)$$

con $\tilde{T}(\rho, \phi) = \begin{pmatrix} \rho\cos(\phi) \\ \rho\sin(\phi) \end{pmatrix}$

$\Rightarrow$

$$|\det(DT(\rho,\phi))| = \left|\det\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} D\tilde{T}(\rho,\phi)\right)\right|$$

$$= \left|\det\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right| \cdot |\det(D\tilde{T}(\rho,\phi))|$$

$$= |ab| \cdot |\rho|$$

$$= ab\rho$$ //

b) Sea $a,b,c>0$ Generalice el resultado anterior para el siguiente cambio de variable

$$T(\rho,\phi,z) = (a\rho\cos(\phi), b\rho\sin(\phi), cz)$$

Sol,

Sea $\tilde{T}(\rho,\phi,z) = (\rho\cos(\phi), \rho\sin(\phi), z)^T$

$$\Rightarrow T = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \cdot \tilde{T}$$

$\Rightarrow$

$$|\det(DT)| = \left|\det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\right| \cdot |\det(D\tilde{T})|$$

$$= |abc| \cdot |\rho| = abc\rho$$

c) Sean $a, b, c > 0$. Considere la siguiente transformación de coordenadas

$$T(r, \theta, \phi) = (ar\cos(\phi)\sin(\theta), br\sin(\phi)\sin(\theta), cr\cos(\theta))$$

Muestre que se cumple:

$$|\mathrm{Det}(DT(r,\theta,\phi))| = abcr^2\sin(\theta)$$

Sol:

Sea $\tilde{T} = (r\cos(\phi)\sin(\theta), r\sin(\phi)\sin(\theta), r\cos(\theta))$

$$\Rightarrow T = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\tilde{T}$$

$\Rightarrow$

$$|\det(DT)| = \left|\det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\right| \cdot |\det(D\tilde{T})|$$

$$= |abc| \cdot r^2 \sin(\theta)$$

$$= abc\, r^2 \sin(\theta)$$

P3) calcular

a) Calcule para $\alpha \in (0, \frac{\pi}{2})$

$$\int_0^\infty \int_0^\infty \exp(-(x^2+2xy\cos(\alpha)+y^2))\,dx\,dy$$

Sol

notamos

$$x^2+2xy\cos(\alpha)+y^2$$

$$= x^2+2xy\cos(\alpha)+\cos^2(\alpha)y^2+\operatorname{sen}^2(\alpha)y^2$$

$$= (x+\cos(\alpha)y)^2+\operatorname{sen}^2(\alpha)y^2$$

$\therefore$ queremos [en nuestros sueños]

$$\rho\cos(\phi) = x+\cos(\alpha)y$$

$$\rho\operatorname{sen}(\phi) = \operatorname{sen}(\alpha)y$$

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$\Rightarrow X = \rho\cos(\phi) - \cos(\alpha)Y$

$Y = \rho\,\text{sen}(\phi)\cdot\frac{1}{\text{sen}(\alpha)}$

$\Rightarrow Y = \frac{\rho\,\text{sen}(\phi)}{\text{sen}(\alpha)}$

$X = \rho\cos(\phi) - \rho\,\text{sen}(\phi)\cot(\alpha)$

así hacemos el CV

$T(\rho,\phi) = \left(\rho\cos(\phi) - \rho\,\text{sen}(\phi)\cot(\alpha), \frac{\rho\,\text{sen}(\phi)}{\text{sen}(\alpha)}\right)$

$$\frac{\partial T}{\partial \rho} = \begin{pmatrix} \cos(\phi) - \text{sen}(\phi)\cot(\alpha) \\ \frac{\text{sen}(\phi)}{\text{sen}(\alpha)} \end{pmatrix}$$

$$\frac{\partial T}{\partial \phi} = \begin{pmatrix} -\rho\,\text{sen}(\phi) - \rho\cos(\phi)\cot(\alpha) \\ \frac{\rho\cos(\phi)}{\text{sen}(\alpha)} \end{pmatrix}$$

$$J_T = \begin{pmatrix} \cos(\phi) - \operatorname{sen}(\phi)\cot(\alpha) & -\rho\operatorname{sen}(\phi) - \rho\cos(\phi)\cot(\alpha) \\ \dfrac{\operatorname{sen}(\phi)}{\operatorname{sen}(\alpha)} & \dfrac{\rho\cos(\phi)}{\operatorname{sen}(\alpha)} \end{pmatrix}$$

$\det J_T$

$$= \frac{\rho\cos^2(\phi)}{\operatorname{sen}(\alpha)} - \rho\operatorname{sen}(\phi)\frac{\cot(\alpha)\cos(\alpha)}{\operatorname{sen}(\alpha)}$$

$$+ \frac{\rho\operatorname{sen}^2(\phi)}{\operatorname{sen}(\alpha)} + \rho\operatorname{sen}(\alpha)\frac{\cos(\alpha)\cot(\alpha)}{\operatorname{sen}(\alpha)}$$

$$= \frac{\rho}{\operatorname{sen}(\alpha)}$$

así como $\alpha \in (0, \pi/2)$

$$\Rightarrow |\det J_T| = \frac{\rho}{\operatorname{sen}(\alpha)}$$

$$I = \int \int \exp(-\rho^2) \frac{\rho}{\operatorname{sen}(\alpha)} \, d\phi \, d\rho$$

Para los límites notamos

$$y \geq 0 \Rightarrow \frac{\rho \operatorname{sen}(\phi)}{\operatorname{sen}(\alpha)} \geq 0 \Rightarrow \rho \in (0, +\infty)$$
$$\phi \in (0, \pi)$$

$$x \geq 0 \Rightarrow \rho \cos(\phi) \geq \rho \operatorname{sen}(\phi) \cot(\alpha)$$

$$\Rightarrow \cos(\phi) \underset{*}{\geq} \operatorname{sen}(\phi) \cot(\alpha)$$

al tener $* \Rightarrow \phi \in [0, \pi/2]$

$$\Rightarrow \cot(\phi) \geq \cot(\alpha)$$

$\Rightarrow \alpha \geq \phi$ porque cot es decreciente en $[0, \pi/2]$

así los límites quedan:

$$I = \int_0^{+\infty} \int_0^{\alpha} \frac{e^{-\rho^2}\rho}{\operatorname{sen}(\alpha)} \, d\theta \, d\rho$$

$$= \frac{\alpha}{\operatorname{sen}(\alpha)} \int_0^{+\infty} \rho e^{-\rho^2} \, d\rho$$

$$= \frac{\alpha}{\operatorname{sen}(\alpha)} \left. -\frac{e^{-\rho^2}}{2} \right|_0^{+\infty}$$

$$= \frac{\alpha}{2\operatorname{sen}(\alpha)}$$

b) $\displaystyle\int_V \frac{xyz}{x^2+y^2} \, dx\,dy\,dz$

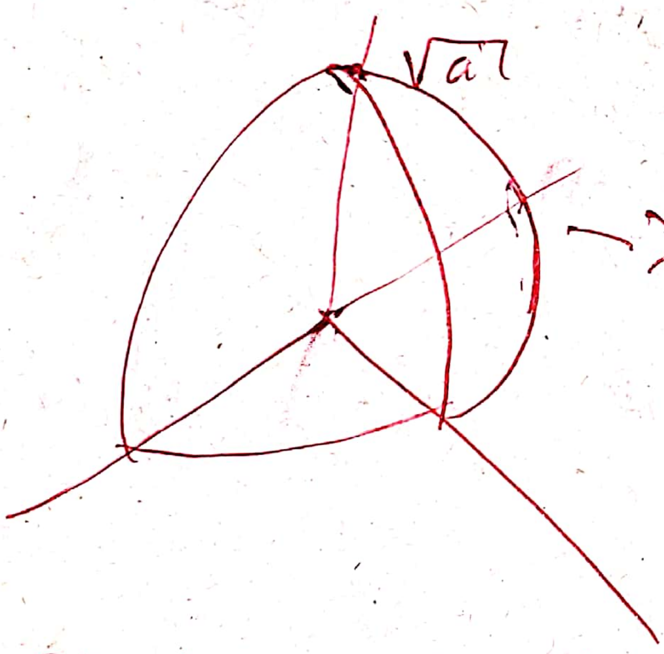
donde V es el dominio restringido por la superficie $(x^2+y^2+z^2)^2 = a^2$ y el plano $z=0$

Sol

notamos que

$$(x^2+y^2+z^2)^2 = a^2$$

$$\Rightarrow x^2+y^2+z^2 = a$$



→ semiesfera de radio $\sqrt{a}$

así haciendo la transformación a [illegible]

$\rho \in (0, \sqrt{a})$

$\phi \in [0, 2\pi)$ $\phi \in [0, \pi/2)$

asi sea

$$x = \pi \cos(\phi) \operatorname{sen}(\theta)$$
$$y = \pi \operatorname{sen}(\phi) \operatorname{sen}(\theta)$$
$$z = \pi \cos(\theta)$$

$$\to \int_V \frac{xyz}{x^2+y^2}\, dx\, dy\, dz$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{a}} \frac{\pi\cos(\phi)\operatorname{sen}(\theta)\, \pi\operatorname{sen}(\phi)\operatorname{sen}(\theta)\, \pi\cos(\theta)\, \pi^2 \cancel{\operatorname{sen}(\theta)}}{\pi^2 \operatorname{sen}^2(\theta)}\, d\pi\, d\theta\, d\phi$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{a}} \pi^2 \operatorname{sen}(\theta) \cos(\theta) \cos(\phi) \operatorname{sen}(\phi)\, d\pi\, d\theta\, d\phi$$

$$= \left(\underbrace{\int_0^{2\pi} \operatorname{sen}(\phi)\cos(\phi)\, d\phi}_{\to 0} \right) \cdot \left[\int_0^{\frac{\pi}{2}} \operatorname{sen}(\theta)\cos(\theta)\, d\theta \right] \cdot \left[\int_0^{\sqrt{a}} \pi^2\, d\pi \right]$$

$$= 0 \,//$$

$\int_D (x^4+y^4)\,dx\,dy\,dz$ com

$D=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2-1\leq z\leq 1\}$

Sol

sea $x=\rho\cos(\phi)$, $y=\rho\,\text{sen}(\phi)$

$\rightarrow \phi\in[0,2\pi)$

$\rho^2-1\leq z\leq 1$

$\Rightarrow \rho^2-1\leq 1$

$\Rightarrow \rho\leq\sqrt{2}$

$\Rightarrow \rho\in(0,\sqrt{2})$

así

$$\int_D (x+y)\,dx\,dy\,dz=\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{\rho^2-1}^{1}\rho^4[\cos^4(\phi)+\text{sen}^4(\phi)]\rho\,dz\,d\rho\,d\phi$$

$$=\int_0^{2\pi}\int_0^{\sqrt{2}}\rho^5(2-\rho^2)[\cos^4(\phi)+\text{sen}^4(\phi)]\,d\rho\,d\phi$$

$$= \int_0^{2\pi} [\cos^4(\phi) + \text{sen}^4(\phi)] \left(\frac{2\rho^6}{6} - \frac{\rho^8}{8} \right)\Big|_0^{\sqrt{2}} d\phi$$

$$= \int_0^{2\pi} [\cos^4(\phi) + \text{sen}^4(\phi)] \cdot \frac{2}{3}\, d\phi$$

$$= \frac{2}{3} \int_0^{2\pi} [\cos^4(\phi) + \text{sen}^4(\phi)\, d\phi.$$

$$= \frac{2}{3} \cdot \frac{3\pi}{2} = \pi$$

~~d) Sea $f: \mathbb{R} \to \mathbb{R}$ una función integrable considere la integral:~~

$$\cancel{\int_{-a}^{a} \int_{-a}^{a} \int_{\mathbb{R}} f(x-y-z)\, dx\, dy\, dz}$$

~~¿es posible hacer un cambio de~~

d) Muestre

$$I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy = \frac{\pi a}{4}$$

<u>Sol</u>

Sea $x = \rho\cos(\phi)$, $y = \rho\,\mathrm{sen}(\phi)$

$\rightarrow$ $h(\rho,\phi) = (\rho\cos(\phi), \rho\,\mathrm{sen}(\phi))$

$$Jh = \begin{pmatrix} \cos(\phi) & -\rho\,\mathrm{sen}(\phi) \\ \mathrm{sen}(\phi) & \rho\cos(\phi) \end{pmatrix}$$

$\det(Jh) = \rho$

ahora

$$0 \leq \rho\,\mathrm{sen}(\phi) \leq a$$

$$\rho\,\mathrm{sen}(\phi) = y \leq \rho\cos(\phi) \leq a$$

$\Rightarrow \quad 0 \leq \rho\, \mathrm{sen}(\phi) \leq a$

$\rho\, \mathrm{sen}(\phi) \leq \rho \cos(\phi) \leq a$

$\mathrm{sen}(\phi) \leq \cos(\phi)$

$\therefore \phi \in [0, \pi/4]$

$\rightarrow \cancel{0 \leq \rho \leq a/\mathrm{sen}(\phi}$

$\rho\, \mathrm{sen}(\phi) \leq \rho \cos(\phi) \leq a$

$\Rightarrow \quad \rho \cos(\phi) \leq a$

$\Rightarrow \rho \leq a/\cos(\phi)$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \int_0^{a/\cos(\phi)} \frac{\rho \cos(\phi) \rho\, d\rho\, d\phi}{\rho^2}$$

$$= \int_0^{\frac{\pi}{4}} \cos(\phi) \cdot a/\cos(\phi)\, d\phi = \frac{a\pi}{4} \; //$$

e) considere la función

$f(x,y,z) = x^2+4y^2+9z^2$ encuentre el valor de:

$$I = \int_R e^{\sqrt{f(x,y,z)}}\, dx\,dy\,dz$$

donde $R = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) \leq 16\}$

Sol

Sea

$$T(r,\theta,\phi) = \left(r\cos(\phi)\operatorname{sen}(\theta), \frac{r}{2}\operatorname{sen}(\phi)\operatorname{sen}(\theta), \frac{r}{3}\cos(\theta)\right)$$

$\Rightarrow$ Por P_2. c Sabemos

$$|\operatorname{Det}(DT)| = \frac{1}{6}r^2\sin(\theta)$$

Por otro lado

$$x^2 + 4y^2 + 9z^2 = r^2$$

$$\Rightarrow R = \{(r, \theta, \phi) \mid r^2 \leq 16\}$$

$$r \in [0, 4)$$
$$\theta \in [0, \pi)$$
$$\phi \in [0, 2\pi)$$

así

$$I = \int_0^4 \int_0^{2\pi} \int_0^{\pi} e^{r} \frac{1}{6} r^2 \sin(\theta)\, r \, d\theta \, d\phi \, dr$$

$$= \frac{1}{6} \int_0^4 e^{r} r^3 \int_0^{2\pi} \int_0^{\pi} \sin(\theta) \, d\theta \, d\phi \, dr$$

$$= \frac{1}{6} \int_0^4 e^{r} r^3 \int_0^{2\pi} -\cos(\theta) \Big|_0^{\pi} d\phi \, dr$$

$$= \frac{1}{6} \int_0^4 e^{r} r^3 \int_0^{2\pi} 2 \, d\phi \, dr$$

$$= \frac{4\pi}{6} \int_0^4 e^{r} r^3 \, dr$$

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$$= \frac{2\pi}{3} \int_0^4 e^r r^3 \, dr$$

$$= \frac{2\pi}{3} \left[e^r (r^3 - 3r^2 + 6r - 6) \right]_0^4$$

$$= \frac{2\pi}{3} \left[e^4 (4^3 - 3 \cdot 4^2 + 6 \cdot 4 - 6) + 6 \right]$$

$$= \frac{2\pi}{3} \left[34 e^4 + 6 \right] //$$

f) sea $g: \mathbb{R}^3 \to \mathbb{R}^3$ una función definida como

$$g(x, y, z) = (3y + 4z, 2x - 3z, x + 3y)$$

Sea $S = [0,1]^3$. Encuentre $\alpha \in \mathbb{R}$ T.q

$$\int_{g(S)} (2x + y - 2z) \, dx\,dy\,dz = \alpha \int_S z \, dx\,dy\,dz$$

Sol) sea $f(x,y,z) = 2x+y-2z$

notamos

$(f\circ g) = 2[3y+4z]+2x-3z-2[x+3y]$

$= \cancel{6y} + 8z + \cancel{2x} - 3z - \cancel{2x} - \cancel{6y}$

$= 5z$

$\therefore$ si $|Det(g)| = cte = \beta$ entonces usando TCV obtenemos

$$\int_{g(S)} (2x+y-2z)\,dx\,dy\,dz$$

$$= \int_S 5z \cdot \beta \, dx\,dy\,dz$$

$$= 5\beta \int_S z\,dx\,dy\,dz \underset{\substack{\downarrow \\ \text{solo si}}}{=} \alpha \int_S z\,dx\,dy\,dz$$

$5\beta = \alpha$

$\therefore$ Calculemos Dg

$$\frac{\partial g}{\partial x} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad \frac{\partial g}{\partial y} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$

$$\frac{\partial g}{\partial z} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$$

$$\Rightarrow Dg = \begin{pmatrix} 0 & 3 & 4 \\ 2 & 0 & -3 \\ 1 & 3 & 0 \end{pmatrix}$$

$$\Rightarrow \det Dg = \begin{vmatrix} 0 & 3 & 4 \\ 2 & 0 & -3 \\ 1 & 3 & 0 \end{vmatrix}$$

$$= 0 \cdot \begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 3 \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix}$$

$$+ 4 \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} = -3\,[3] + 4 \cdot [2 \cdot 3]$$

$$= -9 + 24 = 15$$

$\Rightarrow \beta = 15 \qquad \alpha = 5 \cdot 15$

$\Rightarrow \alpha = 75$

P4) El objetivo de este problema es calcular la integral:

$$I = \int_0^{+\infty} \int_x^{+\infty} e^{-(x-y)^2} \cdot \sin(x^2+y^2) \frac{x^2-y^2}{(x^2+y^2)^2} \, dy\, dx$$

Para esto usaremos herramientas de ecuaciones diferenciales. Con ese proposito en mente siga los siguientes pasos:

a) escriba la integral en alguna coordenada conveniente

Sol)

hacemos el cambio a polares:

$x = \rho\cos(\phi)$, $y = \rho\,\text{sen}(\phi)$

$\rho \in [0, +\infty)$, $\phi \in [0, 2\pi)$ [a priori]

ahora

$0 \leq x \leq +\infty$

$x \leq y \leq +\infty$

$\Rightarrow 0 \leq \rho\cos(\phi)$ (1)

$\rho\cos(\phi) \leq \rho\,\text{sen}(\phi)$ (2)

(1) $\Rightarrow 0 \leq \cos(\phi)$

$\therefore \phi \in [0, \frac{\pi}{2}]$

ahora de (2)

$\cos(\phi) \leq \text{sen}(\phi) \Rightarrow \phi \in [\frac{\pi}{4}, \frac{\pi}{2}]$

asi observamos

$$-(x-y)^2 = -(\rho\cos(\phi) - \rho\,\mathrm{sen}(\phi))^2$$

$$= -\rho^2[1 - \mathrm{sen}(2\phi)]$$

por otro lado

$$\sin^2(x^2+y^2) = \sin^2(\rho^2)$$

$$(x^2-y^2) = \rho^2\cos^2(\phi) - \rho^2\mathrm{sen}^2(\phi)$$

$$= \rho^2\cos(2\phi)$$

$$(x^2+y^2)^2 = \rho^4$$

asi

$$I = \int_0^{+\infty}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-\rho^2[1-\mathrm{sen}(2\phi)]} \cdot \frac{\rho^2\cos(2\phi)}{\rho^4} \cdot \sin^2(\rho^2)\,\rho\, d\phi\, d\rho$$

$$= \int_0^{+\infty} \frac{\sin^2(\rho^2)}{\rho} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-\rho^2[1-\text{sen}(2\phi)]} \cdot \cos(2\phi)\, d\phi\, d\rho$$

$$= \int_0^{+\infty} \frac{\sin^2(\rho^2)}{\rho} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2\rho^2} \frac{d}{d\phi} e^{-\rho^2[1-\text{sen}(2\phi)]}\, d\phi\, d\rho$$

$$= \int_0^{+\infty} \frac{\sin^2(\rho^2)}{2\rho^3} \left. e^{-\rho^2[1-\text{sen}(2\phi)]} \right|_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{2}} d\rho$$

$$= \int_0^{+\infty} \frac{\sin^2(\rho^2)}{2\rho^3} [e^{-\rho^2}-1]\, d\rho$$

$$= \int_0^{+\infty} \frac{\sin^2(\lambda)}{4\lambda^2} [e^{-\lambda}-1]\, d\lambda$$

$\downarrow$

$\lambda = \rho^2 \rightarrow d\lambda = 2\rho\, d\rho$

b]

considere la siguiente función:

$$f(x) = \frac{\sin^2(ax)}{x^2}$$

Muestre que f es de orden exponencial

obs] g es d orden exponencial si $\exists \alpha \in \mathbb{R}$
$g: [0,\infty) \to \mathbb{R}$ T.q $|g(x)| \leq M e^{\alpha x}$ y g es continua por pedazos [se dise $g \in \mathcal{E}_\alpha$]

Sol]

observamos que

$$\lim_{x \to 0^+} f(x) = a^2 \quad \therefore f \text{ es continua en } [0,+\infty)$$

$\Rightarrow f$ es continua por pedazos. Por otro lado

$$|f(x)| \leq \frac{1}{x^2} \qquad \forall x \in [1,+\infty)$$

$$\leq e^{x} \qquad \forall x \in [1,+\infty)$$

como f es continua en $[0,1]$
$\Rightarrow \exists \bar{x} \in [0,1]$ T.q
$$|f(x)| \leq |f(\bar{x})|$$

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$\Rightarrow |f(\bar{x})| \leq |f(\bar{x})| e^{x} \quad \forall x \in [0, +\infty)$

$\Rightarrow |f(x)| \leq |f(\bar{x})| e^{x} \quad \forall x \in [0, 1]$

$\Rightarrow |f(x)| \leq |f(\bar{x})| e^{x} + e^{x} \quad \forall x \in [0, +\infty)$

$= [1 + f(\bar{x})] e^{x}$

$= M e^{x}$

$\therefore f \in \mathcal{E}_1$

c) Construya EDO para la transformada de LaPlace de la función f de la parte b. Para esto recuerde que

$$\mathcal{L}[x^n f(x)](s) = (-1)^n \frac{d^n}{ds^n} \mathcal{L}[f(x)](s)$$

Sol,

$$f(x) = \frac{\sin^2(ax)}{x^2}$$

$\Rightarrow x^2 f(x) = \sin^2(ax)$

$\Rightarrow \mathcal{L}[x^2 f(x)] = \mathcal{L}[\sin^2(ax)]$

$$\Rightarrow \frac{d^2}{ds^2} \mathcal{L}[f(x)] = \mathcal{L}[\sin^2(ax)]$$ //

d) Usando lo anterior considere que la siguiente transformada de Laplace tiene un valor conocido

$$\mathcal{L}[\sin^2(ax)](s) = \frac{2a^2}{s(s^2+4a^2)}$$

Y resuelva la edo encontrada en la parte c.

Indicación: Recuerde que si $g \in \mathcal{C}_\alpha$ entonces la transformada existe y cumple $\lim_{s\to\infty} \mathcal{L}[g(x)](s) = 0$ Hint: Calcule primitivas, encuentre las constantes una vez consiga la transformada de f.

Sol/

$$\frac{d^2}{ds^2} \mathcal{L}[f(x)](s) = \frac{2a^2}{s(s^2+4a^2)}$$

$$\Rightarrow \frac{d}{ds}\mathcal{L}[f(x)](s) = \int \frac{2a^2}{s(s^2+4a^2)}\, ds$$

$$= \int \frac{1}{2s} - \frac{s}{2(s^2+4a^2)}$$

$$= \frac{1}{2}\ln(s) - \frac{1}{4}\ln(s^2+4a^2) + C$$

$$= \frac{1}{4}\left[2\ln(s) - \ln(s^2+4a^2)\right] + C$$

$$= \frac{1}{4}\left[\ln(s^2) - \ln(s^2+4a^2)\right] + C$$

$$= \frac{1}{4}\ln\left(\frac{s^2}{s^2+4a^2}\right) + C$$

$$\Rightarrow \frac{d}{ds}\mathcal{L}[f(t)](s) = \frac{1}{4}\ln\left(\frac{s^2}{s^2+4a^2}\right) + C$$

$$\Rightarrow \mathcal{L}[f(t)](s) = \frac{1}{4}\int \ln\left(\frac{s^2}{s^2+4a^2}\right) + Cs + D$$

$$\int \ln\left(\frac{s^2}{s^2+4a^2}\right) ds = \cancel{\ }$$

$$U = \ln\left(s^2/s^2+4a^2\right) \rightarrow dU = \frac{1}{s^2/s^2+4a^2} \cdot \left[\frac{2s[s^2+4a^2] - 2s.s^2}{[s^2+4a^2]^2}\right]$$

$$dV = 1 \longrightarrow V = s.$$

$$= s\ln\left(\frac{s^2}{s^2+4a^2}\right) - \int \cancel{s} \cdot \frac{\cancel{s^2+4a^2}}{\cancel{s^2}} \cdot \left[\frac{2\cancel{s}\cdot 4a^2}{[s^2+4a^2]^{\cancel{2}}}\right] ds$$

$$= s\ln\left(s^2/s^2+4a^2\right) - \int \frac{8a^2}{s^2+4a^2}\, ds$$

$$= s\ln\left(s^2/s^2+4a^2\right) - 8a^2 \int \frac{1}{s^2+4a^2}\, ds$$

$$= s\ln\left(s^2/s^2+4a^2\right) - \frac{8a^2}{4a^2} \int \frac{1}{\left(\frac{s}{2a}\right)^2+1}\, ds$$

$$= s\ln\left(s^2/s^2+4a^2\right) - 2\int \frac{1}{\left(\frac{s}{2a}\right)^2+1}\, ds$$

$$= s\ln\left(\frac{s^2}{s^2+4a^2}\right) - 2 \cdot \arctan\left(\frac{s}{2a}\right) \cdot 2a$$

$$= s\ln\left(\frac{s^2}{s^2+4a^2}\right) - 4a \cdot \arctan\left(\frac{s}{2a}\right)$$

así

$$\mathcal{L}[f(x)](s) = \frac{s}{4}\ln\left(\frac{s^2}{s^2+4a^2}\right) - a \cdot \arctan\left(\frac{s}{2a}\right) + Cs + D$$

ahora falta encontrar las constantes

como $f \in \mathcal{C}_1 \Rightarrow \lim_{s\to\infty} \mathcal{L}[f(x)](s) = 0$

asi

$$\lim_{s\to\infty} \frac{s}{4}\ln\left(\frac{s^2}{s^2+4a^2}\right) - a\cdot\arctan\left(\frac{s}{2a}\right) + Cs + D = 0$$

$$\lim_{s\to\infty} s\cdot\ln\left(\frac{s^2}{s^2+4a^2}\right) = 0 \quad [\text{verificar!}]$$

$\Rightarrow$ como $\lim_{s\to\infty} -a\arctan\left(\frac{s}{2a}\right) = -\frac{a\pi}{2}$

$\Rightarrow C=0$ para que el limite exista

$\therefore -\frac{a\pi}{2} + D = 0$

$\Rightarrow D = a\pi/2$

$$\therefore \mathcal{L}[f(x)](s) = \frac{s}{4}\ln\left(\frac{s^2}{s^2+4a^2}\right) - a\cdot\arctan\left(\frac{s}{2a}\right) + a\pi/2$$

e) calcule el resultado dela integral com el resultado anterior notamos que

$$I = \frac{1}{4}\int_0^{+\infty} \frac{\sin^2(\lambda)}{\lambda^2}\,[e^{-\lambda}-1]\,d\lambda$$

$$= \frac{1}{4}\left[\int_0^{+\infty} \frac{\sin^2(\lambda)\,e^{-\lambda}}{\lambda^2} - \int_0^{+\infty} \frac{\sin^2(\lambda)}{\lambda^2}\,d\lambda\right]$$

$$= \frac{1}{4}\left[\mathcal{L}[F(\lambda)](1) - \mathcal{L}[F(\lambda)](0)\right], \text{ cuando } a=1$$

así

$$\mathcal{L}[F(\lambda)](s) = \frac{s}{4}\ln\left(\frac{s^2}{s^2+4}\right) - \arctan\left(\frac{s}{2}\right) + \frac{\pi}{2}$$

$$\mathcal{L}[F(\lambda)](1) = \frac{1}{4}\ln\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{2}\right) + \frac{\pi}{2}$$

$$\mathcal{L}[F(\lambda)](0) = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{1}{4}\left[\frac{1}{4}\ln(1/5) - \arctan(1/2)\right]$$

$$= \frac{1}{16}\ln(1/5) - \frac{\arctan(1/2)}{4}$$