P1) Calculo Diferencial
Regla de la cadena

Las coordenadas elípticas se definen por las ecuaciones $X(u,v) = \operatorname{senh}(u)\cdot \operatorname{sen}(v)$ $Y(u,v) = \cosh(u)\cdot\cos(v)$. Sea $f:\mathbb{R}^2 \to \mathbb{R}$ una función de clase $\mathcal{C}^2$. Su representación elíptica ~~de~~ es la función:

$$g(u,v) = f(X(u,v), Y(u,v))$$

a) Demuestre que

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial X}\cdot \cosh(u)\cdot \operatorname{sen}(v) + \frac{\partial f}{\partial Y}\cdot \operatorname{senh}(u)\cdot \cos(v)$$

$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial X}\cdot \operatorname{senh}(u)\cdot \cos(v) - \frac{\partial f}{\partial Y}\cdot \cosh(u)\cdot \operatorname{sen}(v)$$

Sol|

Regla de la cadena dice:

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$\therefore$ Si mostramos que

$$\frac{\partial x}{\partial u} = \cosh(u) \cdot \operatorname{sen}(v)$$

$$\frac{\partial y}{\partial u} = \operatorname{senh}(u) \cdot \cos(v)$$

$$\frac{\partial x}{\partial v} = \operatorname{senh}(u) \cdot \cos(v)$$

$$\frac{\partial y}{\partial v} = -\cosh(u) \cdot \operatorname{sen}(v)$$

Observamos que

$$x(u,v) = \operatorname{senh}(u) \cdot \operatorname{sen}(v)$$

$$\Rightarrow \frac{\partial x}{\partial u} = \cosh(u) \cdot \operatorname{sen}(v)$$

$$\Rightarrow \frac{\partial x}{\partial v} = \operatorname{senh}(u) \cdot \cos(v)$$

$$y(u,v) = \cosh(u) \cdot \cos(v)$$

$$\Rightarrow \frac{\partial y}{\partial u} = \operatorname{senh}(u) \cdot \cos(v)$$

$$\frac{\partial y}{\partial v} = \cosh(u) \cdot -\operatorname{sen}(v)$$
$$= -\cosh(u) \cdot \operatorname{sen}(v)$$

$\therefore$ se cumple lo pedido.

b) Calcule $\partial^2 g/\partial u^2$, $\partial^2 g/\partial v^2$ y $\partial^2 g/\partial u \partial v$ en términos de las derivadas parciales de f y demuestre que el laplaciano en este sistema de coordenadas se calcula como:

$$\left[\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right]=\frac{1}{\operatorname{sen}^2(v)+\operatorname{senh}^2(u)}\left(\frac{\partial^2 g}{\partial u^2}+\frac{\partial^2 g}{\partial v^2}\right)$$

Soll

como

$$\frac{\partial g}{\partial u}=\frac{\partial f}{\partial x}\cdot\cosh(u)\cdot\operatorname{sen}(v)$$

$$+\frac{\partial f}{\partial y}\cdot\operatorname{senh}(u)\cdot\cos(v)$$

$$\frac{\partial^2 g}{\partial u^2}=\frac{\partial}{\partial u}\left[\frac{\partial f}{\partial x}\cdot\cosh(u)\right]\cdot\operatorname{sen}(v)$$

$$+\frac{\partial}{\partial u}\left[\frac{\partial f}{\partial y}\cdot\operatorname{senh}(u)\right]\cdot\cos(v)$$

me condemos que

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial x}(x(u,v),y(u,v))$$

Lo identificamos

y analogo para $\frac{\partial f}{\partial y}$

$\therefore\ \partial f/\partial x,\ \partial f/\partial y$ depemde de u y v así por la regla de multiplicacióm

$$\frac{\partial}{\partial u}\left[\frac{\partial f}{\partial x}\cdot \cosh(u)\right]$$

$$= \operatorname{senh}(u)\cdot \frac{\partial f}{\partial x}$$

$$+ \cosh(u)\cdot\left[\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\cdot\frac{\partial y}{\partial u}\right]$$

$$= \operatorname{senh}(u)\cdot \partial f/\partial x$$

$$+ \cosh(u)\left[\partial^2 f/\partial x^2\cdot \cosh(u)\cdot \operatorname{sen}(v)\right.$$

$$\left.+ \partial^2 f/\partial y\partial x\cdot \operatorname{senh}(u)\cdot \cos(v)\right]$$

$$\frac{\partial}{\partial u}\left[\frac{\partial f}{\partial y}\cdot \operatorname{senh}(u)\right]$$

$$= \cosh(u)\cdot \partial f/\partial y$$

$$+ \operatorname{senh}(u)\left[\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial u}\right]$$

$= \cosh(u)\, \partial f/\partial y$

$+ \operatorname{senh}(u) [\partial^2 f/\partial x \partial y \cdot \cosh(u) \operatorname{sen}(v)$

$+ \partial^2 f/\partial y^2 \cdot \operatorname{senh}(u) \cdot \cos(v)]$

$\frac{\partial^2 g}{\partial u^2} = \operatorname{sen}(v) \cdot \Big[\operatorname{senh}(u)\, \partial f/\partial x$

$+ \cosh(u) [\partial^2 f/\partial x^2 \cdot \cosh(u) \operatorname{sen}(v)$

$+ \partial^2 f/\partial y \partial x \cdot \operatorname{senh}(u) \cdot \cos(v)] \Big]$

$+ \cos(v) \Big[\cosh(u)\, \partial f/\partial y +$

$\operatorname{senh}(u) [\partial^2 f/\partial x \partial y \cdot \cosh(u) \cdot \operatorname{sen}(v)$

$+ \partial^2 f/\partial y^2 \cdot \operatorname{senh}(u) \cdot \cos(v)] \Big]$

a su vez

$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x} \cdot \operatorname{senh}(u) \cdot \cos(v)$

$- \frac{\partial f}{\partial y} \cdot \cosh(u) \operatorname{sen}(v)$

así

$$\frac{\partial^2 g}{\partial v^2} = \operatorname{senh}(u) \cdot \frac{\partial}{\partial v}\left[\frac{\partial f}{\partial x} \cdot \cos(v)\right]$$

$$-\cosh(u) \frac{\partial}{\partial v}\left[\frac{\partial f}{\partial y} \cdot \operatorname{sen}(v)\right]$$

así

$$\frac{\partial}{\partial v}\left[\frac{\partial f}{\partial x} \cos(v)\right]$$

$$= -\operatorname{sen}(v)\, \partial f/\partial x$$

$$+ \cos(v) \cdot \left[\partial^2 f/\partial x^2 \cdot \frac{\partial x}{\partial v} + \partial^2 f/\partial y \partial x \cdot \partial y/\partial v\right]$$

$$= -\operatorname{sen}(v)\, \partial f/\partial x$$

$$+ \cos(v)\left[\partial^2 f/\partial x^2 \cdot \operatorname{senh}(u) \cdot \cos(v) + \partial^2 f/\partial y \partial x \cdot -\cosh(u) \cdot \operatorname{sen}(v)\right]$$

$$\frac{\partial}{\partial v}\left[\frac{\partial f}{\partial y}\cdot \operatorname{sen}(v)\right]$$

$$= \cos(v)\cdot \partial f/\partial y$$

$$+ \operatorname{sen}(v)\left[\partial^2 f/\partial x\partial y \cdot \frac{\partial x}{\partial v}\right.$$

$$\left. + \partial^2 f/\partial y^2 \cdot \partial y/\partial v\right]$$

$$= \cos(v)\cdot \partial f/\partial y$$

$$+ \operatorname{sen}(v)\left[\partial^2 f/\partial x\partial y \cdot \operatorname{senh}(u)\cdot\cos(v)\right.$$

$$\left. + \partial^2 f/\partial y^2 \cdot - \cosh(u)\cdot \operatorname{sen}(v)\right]$$

entonces

$$\partial^2 g/\partial v^2 = \operatorname{senh}(u)\cdot\left[-\operatorname{sen}(v)\cdot \partial f/\partial x\right.$$

$$+ \cos(v)\left[\partial^2 f/\partial x^2 \cdot \operatorname{senh}(u)\cdot \cos(v)\right.$$

$$\left.\left. - \partial^2 f/\partial y\partial x \cdot \cosh(u)\cdot \operatorname{sen}(v)\right]\right]$$

$$- \cosh(u)\left[\cos(v)\cdot \partial f/\partial y\right.$$

$$+ \operatorname{sen}(v)\left[\partial^2 f/\partial x\partial y \cdot \operatorname{senh}(u)\cdot \cos(v)\right.$$

$$\left.\left. - \partial^2 f/\partial y^2 \cosh(u)\operatorname{sen}(v)\right]\right]$$

Por otro lado

$$\frac{\partial^2 g}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left[\frac{\partial f}{\partial x} \cdot \operatorname{senh}(u) \cdot \cos(v) - \frac{\partial f}{\partial y} \cdot \cosh(u) \cdot \operatorname{sen}(v)\right]$$

$$= \cos(v) \cdot \frac{\partial}{\partial u}\left[\frac{\partial f}{\partial x} \cdot \operatorname{senh}(u)\right] - \operatorname{sen}(v) \cdot \frac{\partial}{\partial u}\left[\frac{\partial f}{\partial y} \cdot \cosh(u)\right]$$

entonces

$$\frac{\partial}{\partial u}\left[\frac{\partial f}{\partial x} \cdot \operatorname{senh}(u)\right]$$

$$= \cosh(u) \cdot \partial f/\partial x$$

$$+ \operatorname{senh}(u) \cdot \left[\partial^2 f/\partial x^2 \cdot \frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{\partial y}{\partial u}\right]$$

$$= \cosh(u) \cdot \partial f/\partial x$$

$$+ \operatorname{senh}(u) \cdot \left[\frac{\partial^2 f}{\partial x^2} \cdot \cosh(u) \cdot \operatorname{sen}(v) \right.$$

$$\left. + \frac{\partial^2 f}{\partial y \partial x} \cdot \operatorname{senh}(u) \cdot \cos(v) \right]$$

$$\frac{\partial}{\partial u} \left[\frac{\partial f}{\partial y} \cdot \cosh(u) \right]$$

$$= \operatorname{senh}(u) \cdot \frac{\partial f}{\partial y}$$

$$+ \cosh(u) \cdot \left[\frac{\partial^2 f}{\partial x \partial y} \cdot \frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial y}{\partial u} \right]$$

$$= \operatorname{senh}(u) \cdot \partial f/\partial y$$

$$+ \cosh(u) \cdot \left[\frac{\partial^2 f}{\partial x \partial y} \cdot \cosh(u) \cdot \operatorname{sen}(v) \right.$$

$$\left. + \frac{\partial^2 f}{\partial y^2} \cdot \operatorname{senh}(u) \cdot \cos(v) \right]$$

y asi reemplazando obtenemos

$$\frac{\partial^2 g}{\partial u \partial v} = \cos(v) \cdot \Big[\cosh(u) \cdot \partial f/\partial x$$

$$+ \operatorname{senh}(u) \cdot \Big[\frac{\partial^2 f}{\partial x^2} \cdot \cosh(u) \cdot \operatorname{sen}(v)$$

$$+ \frac{\partial^2 f}{\partial y \partial x} \cdot \operatorname{senh}(u) \cdot \cos(v) \Big] \Big]$$

$$- \operatorname{sen}(v) \cdot \Big[\operatorname{senh}(u) \cdot \partial f/\partial y$$

$$+ \cosh(u) \Big[\frac{\partial^2 f}{\partial x \partial y} \cdot \cosh(u) \cdot \operatorname{sen}(v)$$

$$+ \frac{\partial^2 f}{\partial y^2} \cdot \operatorname{senh}(u) \cdot \cos(v) \Big] \Big]$$

de lo anterior calculamos

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}$$

$$= \operatorname{sen}(v) \cdot \Big[\operatorname{senh}(u) \, \partial f/\partial x$$

$$+ \cosh(u) \Big[\partial^2 f/\partial x^2 \cdot \cosh(u) \cdot \operatorname{sen}(v)$$

CONICYT

$+ \partial^2 f/\partial y \partial x \cdot \operatorname{senh}(u) \cdot \cos(v)]]$

$+ \cos(v)[\cosh(u) \cdot \partial f/\partial y +$
$\operatorname{senh}(u)[\partial^2 f/\partial x \partial y \cdot \cosh(u) \cdot \operatorname{sen}(v)$
$+ \partial^2 f/\partial y^2 \cdot \operatorname{senh}(u) \cdot \cos(v)]]$

$+ \operatorname{senh}(u)[- \operatorname{sen}(v) \cdot \partial f/\partial x$
$+ \cos(v)[\partial^2 f/\partial x^2 \cdot \operatorname{senh}(u) \cdot \cos(v)$
$- \partial^2 f/\partial y \partial x \cdot \cosh(u) \cdot \operatorname{sen}(v)]]$

$- \cosh(u)[\cos(v) \partial f/\partial y$
$+ \operatorname{sen}(v)[\partial^2 f/\partial x \partial y \cdot \operatorname{senh}(u) \cdot$
$\cos(v) - \partial^2 f/\partial y^2 \cdot \cosh(u) \cdot \operatorname{sen}(v)]]$

$+ \cancel{\operatorname{sen}(v)[\partial^2 f/\partial x \partial y \cdot \operatorname{senh}(u) \cdot \cos(v)}$
$\cancel{- \partial^2 f/\partial y^2 \cdot \cosh(u) \cdot \operatorname{sen}(v)]}$

$= \frac{\partial f}{\partial x}[\operatorname{sen}(v) \cdot \operatorname{senh}(u) - \operatorname{senh}(u) \operatorname{sen}(v)]$

$+ \frac{\partial f}{\partial y}[\cos(v) \cdot \cosh(u) - \cosh(u) \cdot \cos(v)]$

$+ \frac{\partial^2 f}{\partial x \partial y}[\operatorname{sen}(v) \cdot \cosh(u) \cdot \operatorname{senh}(u) \cdot \cos(v)$
$+ \cos(v) \cdot \operatorname{senh}(u) \cdot \cosh(u) \cdot \operatorname{sen}(v)$

$- \operatorname{senh}(u) \cdot \cos(v) \cdot \cosh(u) \cdot \operatorname{sen}(v)$

$- \cosh(u) \cdot \operatorname{sen}(v) \cdot \operatorname{senh}(u) \cos(v) \Big]$

$+ \dfrac{\partial^2 f}{\partial x^2} \Big[\cosh^2(u) \cdot \operatorname{sen}^2(v)$

$+ \operatorname{sen}^2 h(u) \cdot \cos^2(v) \Big]$

$+ \dfrac{\partial^2 f}{\partial y^2} \Big[\operatorname{sen}^2 h(u) \cdot \cos^2(v)$

$+ \cosh^2(u) \operatorname{sen}^2(v) \Big]$

$$= \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right] \Big[\cosh^2(u) \cdot \operatorname{sen}^2(v) \quad \operatorname{sen}^2 h(u) \cdot \cos^2(v) \Big]$$

notamos

$\operatorname{senh}^2(u) \cos^2(v) + \cosh^2(u) \cdot \operatorname{sen}^2(v)$

$= \operatorname{sen} h^2(u) [1 - \operatorname{sen}^2(v)] + \cosh^2(u) \operatorname{sen}^2(v)$

$= \operatorname{senh}^2(u) - \operatorname{senh}^2(u) \operatorname{sen}^2(v) + \cosh^2(u) \cdot$
$\operatorname{sen}^2(v)$

$= \operatorname{senh}^2(u) + \operatorname{sen}^2(v) [\underbrace{\cosh^2(u) - \operatorname{senh}^2(u)}_{1}]$

$$= \operatorname{senh}^2(u) + \operatorname{sen}^2(v)$$

$\Rightarrow$

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = \left[\operatorname{senh}^2(u) + \operatorname{sen}^2(v)\right] \cdot \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right]$$

$\therefore$

$$\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right] = \frac{1}{\operatorname{senh}^2(u) + \operatorname{sen}^2(v)} \cdot \left[\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}\right]$$ //

P2| Optimización - sin restricciones.

Encuentre y clasifique los puntos críticos de las siguientes funciones, determinando en cada caso si los óptimos encontrados son globales

a) $f(x,y) = 3x^3 - y^2 - 9x - 6y + 1$

Sol

Para buscar los puntos críticos buscamos

$\nabla f = 0$

$\frac{\partial f}{\partial x} = 9x^2 - 9$, $\frac{\partial f}{\partial y} = -2y - 6$

CONICYT

$\Rightarrow \quad 9x^2 - 9 = 0$

$-2y - 6 = 0$

$\Rightarrow \quad x^2 = 1$

$y = -3$

$\Rightarrow (1,-3)$ y $(-1,-3)$

Para ver la naturaleza buscamos su $\mathcal{H}f$

$\dfrac{\partial^2 f}{\partial x^2} = 18x \qquad \dfrac{\partial^2 f}{\partial y^2} = -2$

$\dfrac{\partial^2 f}{\partial y \partial x} = 0$

$$\Rightarrow \mathcal{H}f(x) = \begin{pmatrix} 18x & 0 \\ 0 & -2 \end{pmatrix}$$

Para $(1,-3)$

$\mathcal{H}f = \begin{pmatrix} 18 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \mathcal{H}f$ no es ni def pos.

ni def negativa

$\therefore$ es punto silla

para $(-1,-3)$

$$Hf = \begin{pmatrix} -18 & 0 \\ 0 & -2 \end{pmatrix}$$

Hf es def negativa

$\therefore$ es maximo local

observamos que

$$f(x,0) = 3x^3 - ax + 1$$

$$\therefore \lim_{x \to +\infty} f(x,0) = +\infty$$

$$\lim_{x \to -\infty} f(x,0) = -\infty$$

asi $(-1,-3)$ es solamente maximo local.

b) $g(x,y,z) = x^2 + y^2 + z^2 + xy - xz$

Sol buscamos

$\nabla g = 0$

$$\frac{\partial g}{\partial x} = 2x + y - z$$

$$\frac{\partial g}{\partial y} = 2y + x$$

$$\frac{\partial g}{\partial z} = 2z - x$$

$$\frac{\partial^2 g}{\partial x^2} = 2, \quad \frac{\partial^2 g}{\partial y^2} = 2, \quad \frac{\partial^2 g}{\partial z^2} = 2$$

$$\frac{\partial^2 g}{\partial x \partial z} = -1, \quad \frac{\partial^2 g}{\partial x \partial y} = 1, \quad \frac{\partial^2 g}{\partial z \partial y} = 0$$

asi tenemos que

$2x+y-z=0 \quad (1)$

$2y+x=0 \quad (2)$

$2z-x=0 \quad (3)$

$(2)+(3) \Rightarrow y+z=0.$

$\Rightarrow 2x-z-z=0$

$2[-z]+x=0$

$\Rightarrow 2x=2z$

$x=2z$

$\Rightarrow x=0 \Rightarrow z=0 \Rightarrow y=0$

∴ el único pto crítico es $(0,0,0)$

ahora

$$Hg=\begin{pmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} & \frac{\partial^2 g}{\partial x \partial z} \\ \frac{\partial^2 g}{\partial x \partial y} & \frac{\partial^2 g}{\partial y^2} & \frac{\partial^2 g}{\partial y \partial z} \\ \frac{\partial^2 g}{\partial x \partial z} & \frac{\partial^2 g}{\partial y \partial z} & \frac{\partial^2 g}{\partial z^2} \end{pmatrix}$$

CONICYT

así

$$Hg = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

calculamos sus VP

$$p(\lambda) = |Hg - \lambda I|$$

$$= \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix}$$

$$-1\begin{vmatrix} 1 & 0 \\ -1 & 2-\lambda \end{vmatrix}$$

$$-1\begin{vmatrix} 1 & 2-\lambda \\ -1 & 0 \end{vmatrix}$$

$$= (2-\lambda)^3 - (2-\lambda) - (2-\lambda)$$

$$= (2-\lambda)^3 - 2(2-\lambda)$$

$$= (2-\lambda)\left[(2-\lambda)^2 - 2\right]$$

$$= (2-\lambda)\left[\lambda^2 - 4\lambda + 4 - 2\right]$$

$$= (2-\lambda)\left[\lambda^2 - 4\lambda + 2\right]$$

$$\Rightarrow P(\lambda) = 0$$

$$\lambda = 2$$

$$\lambda^2 - 4\lambda + 2 = 0 \Rightarrow \lambda = \frac{4 \pm \sqrt{16-8}}{2}$$

$$-1\begin{vmatrix} 1 & 0 \\ -1 & 2-\lambda \end{vmatrix}$$

$$-1\begin{vmatrix} 1 & 2-\lambda \\ -1 & 0 \end{vmatrix}$$

$$= (2-\lambda)^3 - (2-\lambda) - (2-\lambda)$$

$$= (2-\lambda)^3 - 2(2-\lambda)$$

$$= (2-\lambda)\left[(2-\lambda)^2 - 2\right]$$

$$= (2-\lambda)\left[\lambda^2 - 4\lambda + 4 - 2\right]$$

$$= (2-\lambda)\left[\lambda^2 - 4\lambda + 2\right]$$

$$\Rightarrow P(\lambda) = 0$$

$$\lambda = 2$$

$$\lambda^2 - 4\lambda + 2 = 0 \Rightarrow \lambda = \frac{4 \pm \sqrt{16-8}}{2}$$

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así $\lambda = 2$

$$\lambda = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2}$$

$$= 2 \pm \sqrt{2}$$

así todos los valores propios son positivos

$\therefore H_f$ es def. positivo

$\Rightarrow (0,0,0)$ es min local

Observamos H_g es def positivo ~~así como matriz~~ como función ~~de~~

$\Rightarrow g$ es convexa

es más es estrictamente convexa $\therefore$

(0,0,0) mínimo global único. //

P3] Leer enunciado en pdf [poco largo]

a) escriba el problema en la forma clasica, es decir,

$$\max \quad f$$

$$\text{sa} \quad g_1 = 0$$

$$g_2 = 0$$

$$g_3 = 0$$

explicitando todas las funciones

Sol)

Por enunciado la

función objetivo es

$f(c_1, c_2, c_3, a_1, a_2)$

$= \ln(c_1) + \beta \ln(c_2) + \beta^2 \ln(c_3)$

Por otro lado queremos que:

$$a_1 + c_1 = w + (1+r) a_0$$

$$a_2 + c_2 = w + (1+r) a_1$$

$$c_3 = w + (1+r) a_2$$

$\therefore$

$g_1(c_1, c_2, c_3, a_1, a_2)$

$= a_1 + c_1 - w - (1+r) a_0$

$g_2(c_1, c_2, c_3, a_1, a_2)$

$= a_2 + c_2 - w - (1+r) a_1$

$g_3(c_1, c_2, c_3, a_1, a_2)$

$= c_3 - w_2(1+r)a_2$

así obtenemos

la forma del problema

de optimización

b) Explicite el

Lagrangiano del

problema en la

forma $L(c_1, c_2, c_3, a_1, a_2, \lambda_1, \lambda_2, \lambda_3)$

<u>sol:</u>

$L(c_1, c_2, c_3, a_1, a_2, \lambda_1, \lambda_2, \lambda_3)$

$= f(c_1, c_2, c_3, a_1, a_2)$

$+ \lambda_1 g_1(c_1, c_2, c_3, a_1, a_2)$

$+ \lambda_2 g_2(c_1, c_2, c_3, a_1, a_2)$

$+ \lambda_3 g_3(c_1, c_2, c_3, a_1, a_2)$

$$= \ln(c_1) + \beta \ln(c_2) + \beta^2 \ln(c_3)$$

$$+ \lambda_1 [a_1 + c_1 - w - (1+r) a_0]$$

$$+ \lambda_2 [a_2 + c_2 - w - (1+r) a_1]$$

$$+ \lambda_3 [c_3 - w - (1+r) a_2] //$$

c) Escriba las condiciones de primer orden con respecto a las variables a_1, a_2 para deducir que los multiplicadores siguen una progresión geométrica

Sol

$$\frac{\partial L}{\partial a_1} = 0 , \quad \frac{\partial L}{\partial a_2} = 0$$

$$\frac{\partial L}{\partial a_1} = \lambda_1 - \lambda_2 (1+r)$$

$$\frac{\partial L}{\partial a_2} = \lambda_2 - \lambda_3 (1+r)$$

$$\lambda_1 - \lambda_2 (1+r) = 0$$

$$\lambda_2 - \lambda_3 (1+r) = 0$$

$$\Rightarrow \quad \lambda_2 = (1+r)\lambda_3$$

$$\Rightarrow \quad \lambda_1 = (1+r)^2 \lambda_3 \quad //$$

d) Escriba las condiciones de primer orden con respecto a c_1, c_2, c_3 para despejar estas variables en términos de λ_3

Sol

$$\frac{\partial L}{\partial c_1} = 0, \quad \frac{\partial L}{\partial c_2} = 0, \quad \frac{\partial L}{\partial c_3} = 0$$

$$\frac{\partial L}{\partial c_1} = \frac{1}{c_1} + \lambda_1$$

$$\frac{\partial L}{\partial c_2} = \frac{\beta}{c_2} + \lambda_2$$

$$\frac{\partial L}{\partial c_3} = \frac{\beta^3}{c_3} + \lambda_3$$

CONICYT

$\therefore \dfrac{1}{c_1} + \lambda_1 = 0 \Rightarrow c_1 = -\dfrac{1}{\lambda_1}$

$c_1 = -\dfrac{1}{(1+r)^2} \cdot \dfrac{1}{\lambda_3}$

$\dfrac{\beta}{c_2} + \lambda_2 = 0$

$\Rightarrow c_2 = -\dfrac{\beta}{\lambda_2} = -\dfrac{\beta}{(1+r)\lambda_3}$

$\rightarrow \dfrac{\beta^2}{c_3} + \lambda_3 = 0$

$\Rightarrow c_3 = -\beta^2/\lambda_3$ //

e) Use las restricciones del problema para despejar los ahorros en términos de λ_3. Calcule $1/\lambda_3$ en término de los datos y encuentre los consumos y ahorros óptimos. ¿Pueden

CONICYT

ser los 3 consumos óptimos iguales? En caso de que su respuesta sea afirmativa diga cuando se cumple.

Sol

Las restricciones son:

$$a_1 + c_1 = w + (1+r)a_0$$

$$a_2 + c_2 = w + (1+r)a_1$$

$$c_3 = w + (1+r)a_2$$

$$\Rightarrow a_1 = w + (1+r)a_0 - c_1$$

$$= w + (1+r)a_0 - \left[-\frac{1}{(1+r)^2} \cdot \frac{1}{\lambda_3} \right]$$

$$= w + (1+r)a_0 + \frac{1}{(1+r)^2} \cdot \frac{1}{\lambda_3}$$

$$\Rightarrow a_2 = w + (1+r)a_1 - c_2$$

$$= w + (1+r)a_1 + \frac{\beta}{(1+r)\lambda_3}$$

CONICYT

$$= w + (1+r)\left[w + (1+r)a_0 + \frac{1}{(1+r)^2} \cdot \frac{1}{\lambda_3} \right]$$

$$+ \frac{\beta}{1+r} \frac{1}{\lambda_3}$$

$$= w + (1+r)w + (1+r)^2 a_0$$

$$+ \frac{1}{1+r} \cdot \frac{1}{\lambda_3} + \frac{\beta}{1+r} \frac{1}{\lambda_3}$$

$$= w[2+r] + (1+r)^2 a_0 + \frac{\beta+1}{1+r} \frac{1}{\lambda_3}$$

Por otro lado

$$c_3 = w + (1+r)a_2$$

asi

$$-\frac{\beta^2}{\lambda_3} = w + (1+r)\left[w[2+r] + (1+r)^2 a_0 + \frac{\beta+1}{1+r} \frac{1}{\lambda_3} \right]$$

$\therefore$

$$-\frac{[\beta^2+\beta+1]}{\lambda_3} = w[1+(1+r)(2+r)] + (1+r)^3 a_0$$

$\therefore$

$$\frac{1}{\lambda_3} = -\frac{1}{\beta^2+\beta+1}\left[(1+(1+r)(2+r))w + (1+r)^3 a_0\right]$$

as;

$$c_1 = -\frac{1}{(1+r)^2}\cdot\frac{1}{\lambda_3}$$

$$= \frac{1}{(1+r)^2}\,\frac{\left[(1+(1+r)(2+r))w + (1+r)^3 a_0\right]}{\beta^2+\beta+1}$$

$$c_2 = -\frac{\beta}{(1+r)}\cdot\frac{1}{\lambda_3}$$

$$= \frac{\beta}{(1+r)}\,\frac{\left[(1+(1+r)(2+r))w + (1+r)^3 a_0\right]}{\beta^2+\beta+1}$$

y

$$c_3 = \beta^2 \frac{\left[(1+(1+r)(2+r))w + (1+r)^3 a_0\right]}{\beta^2+\beta+1}$$

así notamos que

$$c_1 = c_2 = c_3$$

solo si

$$\frac{1}{(1+r)^2} = \frac{\beta}{1+r} = \beta^2$$

así $\beta = \frac{1}{1+r}$ //

Por otro lado

$$a_1 = w + (1+r)a_0 + \frac{1}{(1+r)^2}\frac{1}{\lambda_3}$$

$$= w + (1+r)a_0 + \frac{1}{(1+r)^2}\cdot\frac{1}{\beta^2+\beta+1}\left[w[1+(1+r)(2+r)] + (1+r)^3 a_0\right]$$

$$a_2 = w\,[2+r] + (1+r)^2 a_0$$

$$+ \frac{\beta+1}{1+r} \cdot \frac{-1}{\beta^2+\beta+1} \left[\left(1+(1+r)(2+r)\right) w + (1+r)^3 a_0 \right] //$$

P41 calculo integral

Integral Normal

a)

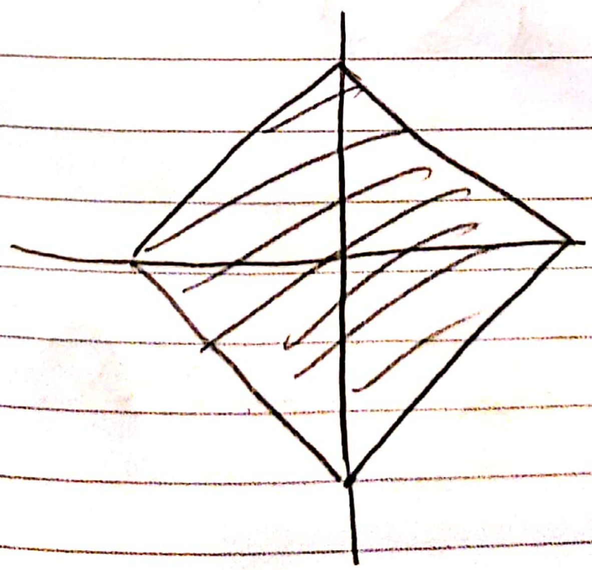
$$\int_G x^2\,dx\,dy$$

Donde $G=\{(x,y)\in\mathbb{R}^2 : |x|+|y|\leq 1\}$

Sol

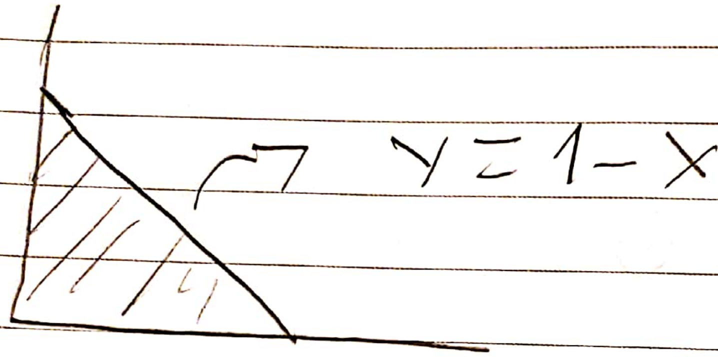
Metodo 1

Grafico



CONICYT

Por simetria basta integrar em Tsup:

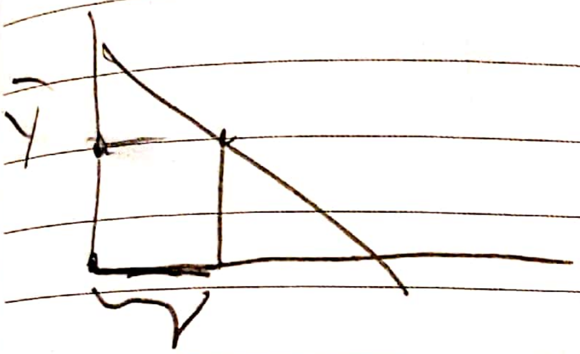


$$\int_{0} x^2\,dx\,dy$$

$$= 4\int_{T_{sup}} x^2\,dx\,dy$$

$$= 4\int_0^1 \int x^2\,dx\,dy$$

ahora falta estudiar los limites deintegración



x se integra de izquierda a derecha: entre 0 y $1-\bar{y}$ así

$$= 4\int_0^1 \int_0^{1-\bar{y}} x^2 \, dx \, dy$$

$$= \frac{4}{3} \int_0^1 (1-y)^3 \, dy$$

$$= \frac{4}{3} \left. \frac{-(1-y)^4}{4} \right|_0^1$$

$$= \frac{1}{3} \; //$$

Metodo 2

ecuaciones

como $|x| + |y| \leq 1$

$$\Rightarrow -(1-|y|) \leq x \leq 1-|y|$$

CONICYT

así

$$\int_G x^2 \, dx \, dy$$

$$= \int_{-1}^{1} \int_{-(1-|y|)}^{1-|y|} x^2 \, dx \, dy$$

paridad

$$= 2 \int_{-1}^{1} \int_{0}^{1-|y|} x^2 \, dx \, dy$$

$$= 2 \int_{-1}^{1} \frac{(1-|y|)^3}{3} \, dy$$

$$= \frac{2}{3} \cdot 2 \int_{0}^{1} (1-|y|)^3 \, dy$$

paridad

$$= \frac{4}{3}\int_0^1 (1-y)^3\,dy$$

$$= \frac{1}{3}$$

$\boxed{b}$

$$\int_1^4 \int_y^4 \int_0^z \frac{z}{x^2+z^2}\,dx\,dz\,dy$$

$$= \int_1^4 \int_y^4 \int_0^1 \frac{z}{(uz)^2+z^2}\, z\,du\,dz\,dy$$

$u = x/z$

CONICYT

$$= \int_1^4 \int_y^4 \int_0^1 \frac{1}{1+u^2}\, du\, dz\, dy$$

$$= \int_1^4 \int_y^4 \arctan(u) \Big|_0^1 dz\, dy$$

$$= \frac{\pi}{4} \int_1^4 \int_y^4 dz\, dy$$

$$= \frac{\pi}{4} \int_1^4 (4-y)\, dy$$

$$= \frac{\pi}{4} \left[4y - \frac{y^2}{2} \right]_1^4$$

$$= \frac{9\pi}{8}$$

15) Calculo integral Fubini

calcule

$$I = \int_{-1}^{0}\int_{-1}^{y} y\sqrt{x^2+y^2}\,dx\,dy$$

Sol

$-1 \leq y \leq 0$

$-1 \leq x \leq y$

$$\int_{-1}^{0}\int_{-1}^{y} y\sqrt{x^2+y^2}\,dx\,dy$$

$$= \int_{-1}^{0}\int_{x}^{0} y\sqrt{x^2+y^2}\,dy\,dx$$

así

$$\int_x^0 y\sqrt{x^2+y^2}\,dy$$

$$= \sqrt{x^2+y^2}\,y^2/2\Big|_x^0 - \frac{1}{2}\int_x^0 \frac{y^3}{\sqrt{x^2+y^2}}$$

$U = \sqrt{x^2+y^2} \rightarrow dU = \dfrac{y}{\sqrt{x^2+y^2}}$

$dV = y \longrightarrow V = y^2/2$

$$= -\frac{\sqrt{2}}{2}|x|\,x^2 - \frac{1}{2}\int_x^0 \frac{y^3}{\sqrt{x^2+y^2}}\,dy$$

$\sqrt{2x^2} = \sqrt{2}\,|x|$

$\neq \sqrt{2}\,x \;;\; x \in [-1,0]$

CONICYT

$|x| = -x \quad , \quad u = x^2 + y^2$

$$= \frac{\sqrt{2}}{2} x^3 - \frac{1}{2} \int_{2x^2}^{x^2} \frac{u - x^2}{\sqrt{u}} \frac{du}{2}$$

$$= \frac{\sqrt{2}}{2} x^3 - \frac{1}{2} \left[\frac{1}{2} \int_{2x^2}^{x^2} \sqrt{u}\, du - \frac{x^2}{2} \int_{2x^2}^{x^2} \frac{1}{\sqrt{u}}\, du \right.$$

$$= \frac{\sqrt{2}}{2} x^3 - \frac{1}{2} \left[-\frac{1}{3}(1 - 2\sqrt{2}) x^3 \right.$$

$$\left. - \frac{x^3}{2} \int_{2x^2}^{x^2} \frac{1}{\sqrt{u}}\, du \right]$$

$$= \frac{\sqrt{2}}{2} x^3 - \frac{1}{2} \left[-\frac{1}{3}(1 - 2\sqrt{2}) x^3 \right.$$

$$\left. - (\sqrt{2} - 1) x^3 \right]$$

$$= \frac{\sqrt{2}}{2} x^3 + \frac{x^3}{2} \left[\frac{1}{3}(1 - 2\sqrt{2}) + \sqrt{2} - 1 \right]$$

$$= \frac{\sqrt{2}}{2} x^3 + \frac{x^3}{2} \frac{1}{3} (\sqrt{2} - 2)$$

$$= x^3 \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{6} - \frac{2}{6} \right]$$

$$= x^3 \left[\frac{4\sqrt{2}}{6} - \frac{2}{6} \right]$$

$$= \frac{x^3}{3} \left[2\sqrt{2} - 1 \right]$$

asi

$$I = \frac{2\sqrt{2} - 1}{3} \int_{-1}^{0} x^3 \, dx$$

$$= \frac{1 - 2\sqrt{2}}{12} \; //$$

P5

solución alternativa

$$\int_{-1}^{0}\int_{-1}^{y} y\sqrt{x^2+y^2}\,dx\,dy$$

$$=\int_{-1}^{0}\int_{x}^{0} y\sqrt{x^2+y^2}\,dy\,dx$$

Fubini

Sea $u = x^2+y^2$

$\rightarrow du = 2y\,dy$

$$=\int_{-1}^{0}\int_{2x^2}^{x^2} \frac{\sqrt{u}}{2}\,du\,dx$$

$$= \frac{1}{2} \int_{-1}^{0} \frac{2}{3} u^{3/2} \Big|_{2x^2}^{x^2} dx$$

$$= \frac{1}{3} \left[\int_{-1}^{0} x^2 |x| - 2^{3/2} x^2 |x| \, dx \right.$$

$$= \frac{1 - 2\sqrt{2}}{3} \left[\int_{-1}^{0} -x^2 |x| \, dx \right]$$

$$= \frac{1 - 2\sqrt{2}}{3} \left[- \int_{-1}^{0} x^3 \, dx \right.$$

$$= \frac{1 - 2\sqrt{2}}{3} \left[- \frac{x^4}{4} \right]_{-1}^{0}$$

CONICYT

$$= \frac{1-2\sqrt{2}}{12}$$

P6) Calculo integral
Teorema de cambio de Variable

Sea $f(x,y,z) = x+y+z$

y $D = \{(x,y,z) \in \mathbb{R}^3 :$

$\sqrt{x^2+y^2} \leq z \leq \sqrt{4-x^2-y^2}$

$$\int_D f$$

Sol

Sea $x = \rho\cos(\phi)$

$y = \rho\,\mathrm{sen}(\phi)$

$z = z$

$\Rightarrow |\mathrm{Det}(Jh)| = \rho$

CONICYT

$$\Rightarrow \rho \leq z \leq \sqrt{4-\rho^2}$$

$$\therefore \rho \in [0,2]$$

así

$$\int_D f = \int_0^2 \int_0^{2\pi} \int_\rho^{\sqrt{4-\rho^2}} (\rho\cos(\phi) + \rho\,\mathrm{sen}(\phi) + z)\,\rho \, dz\, d\phi\, d\rho$$

$$= \int_0^2 \int_0^{2\pi} \int_\rho^{\sqrt{4-\rho^2}} \rho^2 \cos(\phi)\, dz\, d\phi\, d\rho$$

$$+ \int_0^2 \int_0^{2\pi} \int_\rho^{\sqrt{4-\rho^2}} \rho^2 \,\mathrm{sen}(\phi)\, dz\, d\phi\, d\rho$$

$$+ \int_0^2 \int_0^{2\pi} \int_\rho^{\sqrt{4-\rho^2}} \rho z \, dz\, d\phi\, d\rho$$

CONICYT

Scanned by CamScanner

$$= \left[\int_0^{2\pi} \cancel{\cos(\phi)}^{\,0} \right] \cdot \left[\int_0^2 \int_\rho^{\sqrt{4-\rho^2}} \rho^2 \, dz \, d\rho \right]$$

$$+ \left[\int_0^{2\pi} \cancel{\operatorname{sen}(\phi)}^{\,0} \right] \cdot \left[\int_0^2 \int_\rho^{\sqrt{4-\rho^2}} \rho^2 \, dz \, d\rho \right]$$

$$+ \left[\int_0^{2\pi} d\phi \right] \left[\int_0^2 \int_\rho^{\sqrt{4-\rho^2}} \rho z \, dz \, d\rho \right]$$

$$= 2\pi \int_0^2 \rho \cdot \frac{z^2}{2} \Big|_\rho^{\sqrt{4-\rho^2}} d\rho$$

$$= \pi \int_0^2 \rho \left[4 - \rho^2 - \rho^2 \right] d\rho$$

$$= \pi \int_0^2 4\rho - 2\pi \int_0^2 \rho^3 \, d\rho$$

$$= 4\pi \frac{\rho^2}{2} \Big|_0^2 - \frac{2\pi}{4} \rho^4 \Big|_0^2$$

CONICYT

$$= 2\pi 2^2 - \frac{2\pi}{4} 2^4$$

$$= 2^3 \pi - 2^3 \pi$$

$$= 0$$