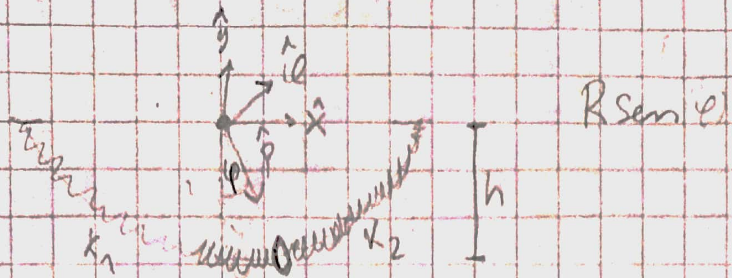
P1

a)



$$x = R\sin(\theta) \quad \wedge \quad y = R\cos(\theta) \quad \wedge \quad z = cte.$$

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(R^2\dot{\theta}^2)$$

$$U = -mgR\cos(\theta) + \frac{1}{2}K_1\left(\frac{\pi R}{2} + \theta R - \frac{\pi R}{2}\right)^2 + \frac{1}{2}K_2\left(\left(\frac{\pi}{2} - \theta\right)R - \frac{\pi R}{2}\right)^2$$

$$U = -mgR\cos(\theta) + \frac{1}{2}K_1(\theta R)^2 + \frac{1}{2}K_2(\theta R)^2$$

$$\Rightarrow L = K - U = \frac{1}{2}m(R\dot{\theta})^2 + mgR\cos(\theta) - \frac{1}{2}(\theta R)^2(K_1 + K_2)$$

b) Usamos la ecuación de Lagrange

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad / \text{ reemplazamos}$$

$$\frac{d(mR^2\dot{\theta})}{dt} + mgR\sin\theta + \theta R^2(K_1 + K_2) = 0$$

$$\Rightarrow mR^2\ddot{\theta} + mgR\sin\theta + \theta R^2(K_1 + K_2) = 0$$

Despejamos $\ddot{\theta}$

$$\ddot{\theta} = -\frac{g}{R}\sin\theta - \frac{\theta}{m}(K_1 + K_2) \quad / \text{ Truco "sucio" mecatrónica}$$

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt}\cdot\frac{d\theta}{d\theta} = \frac{d\dot{\theta}}{d\theta}\cdot\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

$$\Rightarrow \dot{\theta}\,d\dot{\theta} = \left(-\frac{g}{R}\sin\theta - \frac{\theta}{m}(K_1 + K_2)\right)d\theta \quad / \int$$

$$\Rightarrow \boxed{\frac{\dot{\theta}^2}{2} = +\frac{g}{R}\cos\theta - \frac{\theta^2}{2m}(K_1 + K_2)}$$

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c) Para obtener la frecuencia, $\theta \to 0$ y cuando para esto $\sin\theta \approx \theta$ y $\cos\theta \approx 1$.

$$U = -mgR\cos\theta + \frac{1}{2}\theta^2 R^2 (K_1 + K_2)$$

$$\Rightarrow U' = mgR\,\text{sen}\,\theta + \theta R^2 (K_1 + K_2)$$

$$\Rightarrow U'' = mgR\cos\theta + R^2 (K_1 + K_2)$$

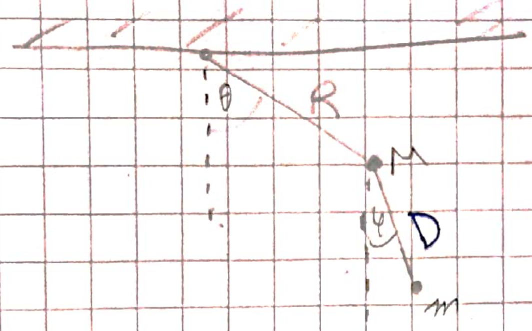
Tenemos que

$$\omega^2 = \frac{U''}{\alpha} \wedge \alpha = mR^2 \Rightarrow \omega^2 = \frac{g}{R}\cos\theta + \frac{(K_1+K_2)}{m}$$

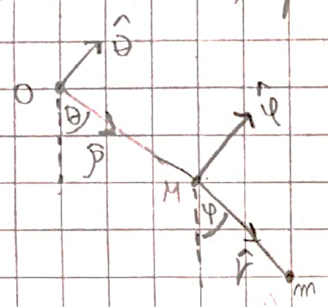
Pero como $\cos\theta \approx 1$ cuando $\theta \to 0$

$$\Rightarrow \boxed{\omega^2 = \frac{g}{R} + \frac{(K_1 + K_2)}{m}}$$

P2



a) Usamos 2 sistemas para describir:



$$r_m = R\hat{\rho} + D\hat{r}$$

$$v_m = R\dot{\theta}\hat{\theta} + D\dot{\varphi}\hat{\varphi} \Rightarrow v_m^2 = R^2\dot{\theta}^2 + D^2\dot{\phi}^2 + 2RD\dot{\theta}\dot{\varphi}\hat{\theta}\hat{\varphi}$$

Como $\hat{\theta} = \hat{y}\sin\theta + \hat{x}\cos\theta \wedge \hat{\varphi} = \hat{y}\sin\varphi + \hat{x}\cos\varphi$

$$\Rightarrow \hat{\theta}\cdot\varphi = \sin\varphi\sin\theta + \cos\varphi\cos\theta = \cos(\theta+\varphi)$$

$$\Rightarrow v_m^2 = R^2\dot{\theta}^2 + D^2\dot{\phi}^2 + 2RD\dot{\theta}\dot{\varphi}\cos(\theta+\varphi)$$

$$\Rightarrow K = \frac{M}{2}R^2\dot{\theta}^2 + \frac{m}{2}\left(R^2\dot{\theta}^2 + D^2\dot{\phi}^2 + 2RD\dot{\theta}\dot{\varphi}\cos(\theta+\varphi)\right)$$

b) $U = -MgR\cos\theta - mg(R\cos\theta + D\cos\varphi)$

c) $L = K - U = \frac{M}{2}R^2\dot{\theta}^2 + \frac{m}{2}(R^2\dot{\theta}^2 + D^2\dot{\phi}^2 + 2RD\dot{\theta}\dot{\varphi}\cos(\theta+\varphi))$
$+ MgR\cos\theta + mg(R\cos\theta + D\cos\varphi)$

Como solo nos interesa θ, derivamos respecto a θ y $\dot{\theta}$.

$$\frac{\partial L}{\partial\dot{\theta}} = (M+m)R^2\dot{\theta} + mRD\dot{\varphi}\cos(\theta+\varphi)$$

$$\frac{\partial L}{\theta} = -mRD\dot{\theta}\dot{\varphi}\sin(\theta+\varphi) - MgR\sin\theta - mgR\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = (M+m)R^2\ddot\theta + mRD\ddot\varphi\cos(\theta+\varphi)$$
$$- mRD\dot\varphi\sin(\theta+\varphi)(\dot\theta+\dot\varphi)$$

Usando la ecuación de Lagrange:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$(M+m)R^2\ddot\theta + mRD\ddot\varphi\cos(\theta+\varphi) - mRD\dot\varphi\sin(\theta+\varphi)\cdot(\cancel{\dot\theta}+\dot\varphi)$$
$$\cancel{+ mRD\dot\theta\dot\varphi\sin(\theta+\varphi)} + (M+m)gR\sin\theta = 0$$

$$\Rightarrow (M+m)R^2\ddot\theta + mRD\ddot\varphi\cos(\theta+\varphi) - mRD\dot\varphi^2\sin(\theta+\varphi)$$
$$+ (M+m)gR\sin\theta = 0$$

Como $\theta \ll 1 \Rightarrow \theta+\varphi \ll 1$. Esto nos dice que

$$\cos(\theta+\varphi) \approx 1 \wedge \sin(\theta+\varphi) \approx 0 \wedge \sin\theta \approx \theta$$

$$\Rightarrow (M+m)R^2\ddot\theta + (M+m)gR\theta = -mRD\ddot\varphi$$

Reescribimos la ecuación y hacemos el cambio
$\varphi = \phi_0\cos(\Omega t)$

$$\ddot\theta + \frac{g}{R}\theta = -\frac{mD}{(M+m)R}\cdot -\phi_0\Omega^2\cos(\Omega t)$$

$$\Rightarrow \ddot\theta + \frac{g}{R}\theta = \frac{mD\phi_0\Omega^2}{(M+m)R}\cos(\Omega t)$$