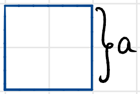
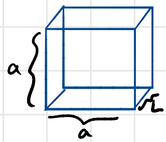
Matriz de inercia de un cuadrado



Hay dos formas de calcular la

Caso 1: Asumir cuerpo en 3D



← Paralelepipedo

Limites de integración

$x \in \left(-\frac{a}{2}, \frac{a}{2}\right) \quad y \in \left(-\frac{a}{2}, \frac{a}{2}\right) \quad z \in \left(-\frac{L}{2}, \frac{L}{2}\right)$

$$I_{xx}^{CM} = \iint \frac{M}{V}(y^2+z^2)\,dx\,dy\,dz$$

$$= \frac{M}{a^2 \cdot L}\int_{-L/2}^{L/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}(y^2+z^2)\,dx\,dy\,dz = \frac{M}{a^2L}\iint (y^2+z^2)\left(\frac{a}{2}-\left(-\frac{a}{2}\right)\right)dy\,dz$$

$$= \frac{M \cdot \cancel{a}}{a^{\cancel{2}}L}\left[\iint y^2\,dy\,dz + \iint z^2\,dy\,dz\right]$$

$$2 \cdot \frac{a^3}{8 \cdot 3} = \frac{a^3}{12}$$

$$= \frac{M}{aL}\left[\int \frac{y^3}{3}\Bigg|_{-a/2}^{a/2} dz + \int z^2 \cdot a\,dz\right]$$

$$= \frac{M}{aL}\left[\frac{a^3}{12}\cdot L + \frac{L^3}{12}\cdot a\right] = M\left[\frac{a^2}{12}+\frac{L^2}{12}\right]$$

Similar pasa con $I_{yy}^{CM} = M\left[\frac{a^2}{12}+\frac{L^2}{12}\right]$; $I_{zz}^{CM} = M\left[\frac{a^2}{12}+\frac{a^2}{12}\right]$

En los otros casos tenemos

$$I_{yz}^{CM} = \frac{M}{a^2L}\left[\iiint yz\,dx\,dy\,dz\right] = \frac{M}{aL}\left[\iint yz\,dy\,dz\right]$$

$$= \frac{M}{aL}\left[\int z\underbrace{\left[\frac{y^2}{2}\Bigg|_{-a/2}^{a/2}\right]}_{0}dz\right] \rightarrow I_{yz}^{CM}=0$$

$I_{xz}^{CM}=0$

$I_{xy}^{CM}=0$

$*$ $\frac{a^2}{2\cdot 4} - \frac{1}{2}\left(\frac{-a}{2}\right)^2$

$\frac{a^2}{8} - \frac{a^2}{8} = 0$

$$I_{ij}^{CM} = \frac{M}{12}\begin{pmatrix} a^2+L^2 & 0 & 0 \\ 0 & a^2+L^2 & 0 \\ 0 & 0 & a^2+a^2 \end{pmatrix}$$

Ahora el caso de un cuadrado es → $L \rightarrow 0$

$$I_{ij}^{CM} = \frac{M}{12}\begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 2a^2 \end{pmatrix}$$

Caso 2: Usar $r^2 = x^2+y^2$ (Aún así se puede tener una matriz de 3x3) $(z=0)$

$$I_{ij}^{CM} = \int \frac{M}{S}\left((x^2+y^2)\delta_{ij} - x_i x_j \right) dS$$

$$I_{xx}^{CM} = \frac{M}{a^2}\int (x^2+y^2-x^2)\, dxdy \rightarrow I_{xx} = \frac{M}{a^2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} y^2\, dxdy$$

$$I_{xx}^{CM} = \frac{M}{a^{\cancel{2}}}\cancel{a}\int_{-a/2}^{a/2} y^2\, dy = \frac{M}{a}\left(\frac{y^3}{3}\Big|_{-a/2}^{a/2} \right)$$

$$= \frac{M}{a}\left(\frac{a^3}{3\cdot 8} + \frac{a^3}{3\cdot 8} \right) = \frac{M}{\cancel{a}}\, a^{\cancel{3}\,2}\, \frac{\cancel{2}}{3\cdot \cancel{8}\,4}$$

$$I_{xx}^{CM} = \frac{M}{12}a^2$$ análogamente $\rightarrow I_{yy}^{CM} = \frac{M}{12}a^2$

$$I_{zz}^{CM} = \frac{M}{a^2}\int (x^2+y^2-\cancel{z^2}^{\,0})\, dxdy = \frac{M}{a^2}\left[\iint x^2\, dxdy + \iint y^2\, dxdy \right]$$

$$= \frac{M}{a^2}\left[\int \frac{a^3}{12} dy + \int y^2\, a\, dy \right]$$

$$= \frac{M}{\cancel{a^2}}\left[\frac{a^{\cancel{4}\,2}}{12} + \frac{a^{\cancel{4}\,2}}{12} \right] = \frac{M}{12}(2a^2)$$

Notemos que los casos I_{xy}^{CM}, I_{yx}^{CM} son analogos al caso anterior

$$I_{xz}^{CM} = \frac{M}{a^2}\int x\cancel{z}^{\,0}\, dxdy = 0$$ igual para $I_{yz}^{CM} = 0$

Podemos ver que se obtiene la misma matriz de inercia