Problema 2

a) Metodo 1, $(\log z)' = \frac{1}{z}$ en $\mathbb{C} \setminus \mathbb{R}^-$

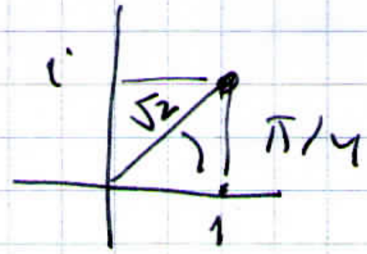
como $[1, 1+i] \subseteq \mathbb{C} \setminus \mathbb{R}^-$, el calculo es

$$\int_{[1,1+i]} \frac{1}{z}\, dz = \log z \Big|_1^{1+i} = \log 1+i - \log 1$$

~~0.5pto~~ 1 pto

$\log 1 = 0 \qquad \log 1+i = \ln|1+i| + i \arg(1+i)$

$|1+i| = \sqrt{2} \qquad \arg(1+i) = \pi/4$



~~0.5pto~~ 0.5pto

así $$\int_{[1,1+i]} \frac{dz}{z} = \ln\sqrt{2} + i\pi/4$$

Otra forma Parametrizar $[1\ 1+i]$

0.6 $\quad z(t) = 1 + it \qquad t \in [0\ 1]$

$dz = i\,dt$

$$\int_{[1\ 1+i]} \frac{1}{z}\, dz = \int_0^1 \frac{i\,dt}{1+it} = \int_0^1 \frac{(1-it)\, i\, dt}{1+t^2}$$

0.4 separar en 2 int. reales

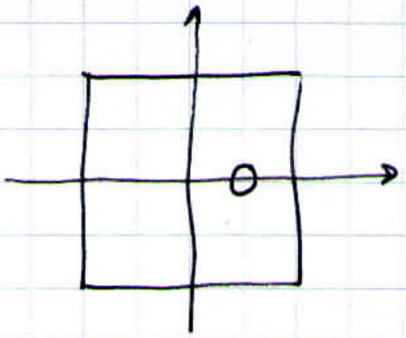
$$= \int_0^1 \frac{i\,dt}{1+t^2} + \int_0^1 \frac{t}{1+t^2}\,dt$$

$$= i \operatorname{atan} t \Big|_0^1 + \int_1^2 \frac{du/2}{u}$$

$1+t^2 = u$
$2t\,dt = du$
$t=0 \Rightarrow u=1$
$t=1 \Rightarrow u=2$

05 $\quad = i\,\pi/4 + \frac{1}{2}\ln(2) \quad$ 05

b)



$$f(z) = \frac{\tan z/2}{(z-1)^2} = \frac{\operatorname{sen} z/2}{\cos(z/2)\,(z-1)^2}$$

El denominador se anula en

0.5 $\quad z = 1 \quad$ y $\quad \frac{z}{2} = \frac{\pi}{2} + k\pi \Leftrightarrow z = \pi + 2k\pi$

solo $z = 1$ se encuentra al interior de γ

Y $z = 1$ es polo de orden 2 $\quad$ 0.5

$$\lim_{z\to 1} (z-1)^2 f(z) = \lim_{z\to 1} \frac{\operatorname{sen} z/2}{\cos z/2} = \tan 1/2 \neq 0$$

así

$$\operatorname{Res}(f;1) = \lim_{z\to 1} \left[(z-1)^2 f(z)\right]'$$

$$= \lim_{z\to 1} \frac{\frac{1}{2}\operatorname{sen}^2 z/2 + \frac{1}{2}\cos^2 z/2}{\cos^2(z/2)}$$

$$= \frac{1}{2}\sec^2(1/2)$$

0.5 calculo residuo

así

$$\int_\gamma \frac{\tan z/2}{(z-1)^2}\,dz = 2\pi i \cdot \frac{1}{2}\sec^2(1/2) = \pi i \sec^2(1/2)$$

0.5 aplicación Teo. Residuo

otra forma, por Formula integral de Cauchy generalizada $f(z) = \tan z/2$ es holomorfa dentro de γ

$$\oint_\gamma \frac{\tan z/2}{(z-1)^2}\, dz = 2\pi i \left(\tan \frac{z}{2}\right)'\Big|_{z=1} = \text{idem}$$ aplicar (1pto)

$$\frac{f(z)}{(z-1)^2} \leftarrow \frac{\text{holomorfa}}{(z-1)^2}$$ (1pto)

c) $$\oint_{|z|=2} \frac{dz}{z^3(z+4)} = 2\pi i \, \text{Res}(f; z=0)$$

$z=0$ es la única singularidad al interior de $|z|=2$ (0,6)

Es polo orden 3 $\quad z^3 \cdot \dfrac{1}{z^3(z+4)} \longrightarrow \dfrac{1}{4}$ (0,4)

$$\text{Res}\{f; z=0\} = \frac{1}{2!} \lim_{z\to 0} \left[z^3 \cdot \frac{1}{z^3(z+4)} \right]''$$

(0,6) calculo residuo

$$= \frac{1}{2} \lim_{z\to 0} \left(\frac{1}{z+4}\right)'' = \frac{1}{2} \lim_{z\to 0} 1\cdot 2\cdot (z+4)^{-3}$$

$$= \frac{1}{2}\, \frac{2}{4^3}$$

así $$\oint_{|z|=2} \frac{dz}{z^3(z+4)} = 2\pi i \cdot \frac{1}{2}\, \frac{2}{4^3}$$

$$= \pi i \cdot \frac{1}{32}$$

aplicar Teo Residuo (0,4)