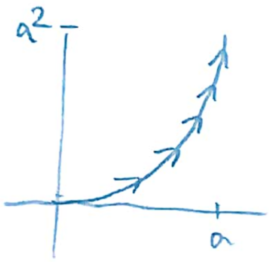
Aux 14

P1 a) $y = x^2$, $x \in [0, a]$ sentido antihorario

↳ Toda función de la forma $y = f(x)$, su parametrización será $\vec{r}(t) = (t, f(t))$, $t \in Dom(f)$

$\Rightarrow$ $\vec{r}(t) = (t, t^2)$ $t \in [0, a]$ y si graficamos



a medida que avanza el x avanza el y en forma antihoraria.

Si queremos sentido horario, nos sirve $\vec{r}_2(t) = (a-t, (a-t)^2)$

b) Segmento de $\vec{p}$ a $\vec{q}$: Usaremos una parametrización que empieza en $\vec{p}$ y termina en $\vec{q}$

$$\Rightarrow \vec{r}(t) = \underbrace{\vec{p} + t\underbrace{(\vec{q} - \vec{p})}_{\text{vector director}}}_{\text{Ec. de la recta}}$$

queremos que en cierto "t_0", $\vec{r}(t_0) = \vec{q}$.

Ese $t_0 = 1$ (Compruebe)

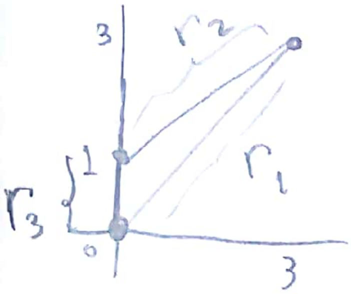
$$\Rightarrow \vec{r}(t) = \vec{p} + t(\vec{q} - \vec{p}) \quad t \in [0, 1]$$

$$= \underbrace{(1-t)\vec{p} + t\vec{q}}$$

A esto se le conoce como Combinación Convexa

P1) c) Triangulo $(0,0), (0,1), (3,3)$

haremos una parametrización por partes



$$\Rightarrow \quad \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

$$\Gamma_1 := \vec{r}_1(t) = (1-t)\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}3\\3\end{pmatrix} \qquad t \in [0,1].$$

$$= t\begin{pmatrix}3\\3\end{pmatrix}$$

$$\Gamma_2 = \vec{r}_2(t) = (1-t)\begin{pmatrix}3\\3\end{pmatrix} + t\begin{pmatrix}0\\1\end{pmatrix} \qquad t \in [0,1].$$

$$\Gamma_3 = \vec{r}_3(t) = (1-t)\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}0\\0\end{pmatrix} \qquad t \in [0,1].$$

$$= (1-t)\begin{pmatrix}1\\0\end{pmatrix}$$

la orientación queda

d) Elipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Coordenadas elípticas $\vec{r}(t) = \begin{pmatrix} 3\cos(t) \\ 2\,\mathrm{Sen}(t) \end{pmatrix}$

e) Casquete unitario $\cap$ $z^2 = x^2 + y^2$

Casquete :: $x^2 + y^2 + z^2 = 1$ $\wedge$ $z^2 = x^2 + y^2$

$\Rightarrow$ $z^2 + z^2 = 1$ $\Rightarrow$ $2z^2 = 1$

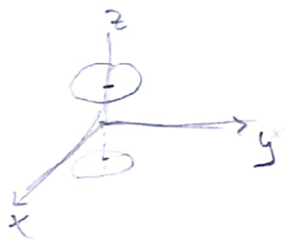
$z = \pm\sqrt{\frac{1}{2}}$

¿Que es $z^2 = x^2 + y^2$?

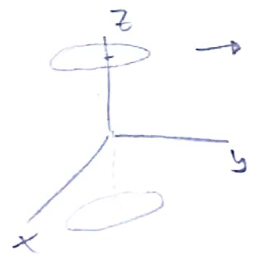
Demos valores de z^2.

Si $z^2 = 0$ $\longrightarrow$

Si $z^2 = 1$ $\longrightarrow$

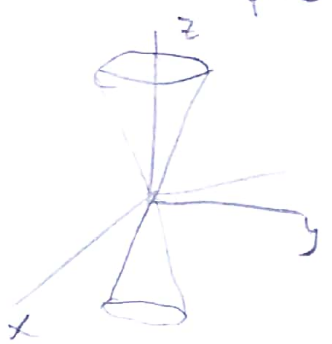


Si $z^2 = 2$ $\longrightarrow$



$\rightarrow$ radio es igual a z^2

Si siguen se daran cuenta que son 2 conos



ahora con coordenadas cilindricas

$x = \rho \cos(t)$, $y = \rho \operatorname{sen}(t)$, $z = z_0 \nearrow$ cte

$\Rightarrow \quad \vec{r}_+(t) = (\rho \cos(t), \rho \operatorname{sen}(t), z_0)$

$= (\rho \cos(t), \rho \operatorname{sen}(t), \frac{1}{\sqrt{2}})$

Notar que $x^2 + y^2 = z^2 \Rightarrow \rho^2 \cos^2(t) + \rho^2 \operatorname{sen}^2(t) = z^2$

$\Rightarrow \quad \rho^2 = z^2$

$\Rightarrow \quad \rho = \pm z = \pm \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \vec{r}_+(t) = \left(\frac{1}{\sqrt{2}} \cos(t), \frac{1}{\sqrt{2}} \operatorname{sen}(t), \frac{1}{\sqrt{2}} \right) := \Gamma_+$

$\wedge \quad \vec{r}_-(t) = -\left(\frac{1}{\sqrt{2}} \cos(t), \frac{1}{\sqrt{2}} \operatorname{sen}(t), \frac{1}{\sqrt{2}} \right) := \Gamma_-$

$\Rightarrow \quad \Gamma = \Gamma_+ \cup \Gamma_-$

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Solución:

1. Consideremos $\vec{r}(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{pmatrix}$ y $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ donde $r_i : [a,b] \to \mathbb{R}$ y $v_i \in \mathbb{R}$, $\forall i$. Entonces

$$\frac{d\vec{r}}{dt}(t) \cdot v = \begin{pmatrix} r_1'(t) \\ r_2'(t) \\ r_3'(t) \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = r_1'(t)v_1 + r_2'(t)v_2 + r_3'(t)v_3$$

   Luego, ocupando que $\vec{r}(a) = P$ y $\vec{r}(b) = Q$.

$$\begin{array}{rcc} \displaystyle\int_a^b \frac{d\vec{r}}{dt}(t) \cdot v dt & = & \displaystyle\int_a^b (r_1'(t)v_1 + r_2'(t)v_2 + r_3'(t)v_3) dt \\ & = & \displaystyle\int_a^b r_1'(t)v_1 dt + \int_a^b r_2'(t)v_2 dt + \int_a^b r_3'(t)v_3 dt \\ & = & \displaystyle v_1 \int_a^b r_1'(t) dt + v_2 \int_a^b r_2'(t) dt + v_3 \int_a^b r_3'(t) dt \\ & = & v_1 \cdot (r_1(b) - r_1(a)) + v_2 \cdot (r_2(b) - r_2(a)) + v_3 \cdot (r_3(b) - r_3(a)) \\ & = & v_1 \cdot v_1 + v_2 \cdot v_2 + v_3 \cdot v_3 \\ & = & v_1^2 + v_2^2 + v_3^2 \\ & = & \|v\|^2 \end{array}$$

2. Consideremos nuevamente la integral $\displaystyle\int_a^b \frac{d\vec{r}}{dt}(t) \cdot v dt$ Usaremos Cauchy-Schwarz sobre $\displaystyle\frac{d\vec{r}}{dt}(t) \cdot v$.

$$\begin{array}{rcc} \displaystyle\int_a^b \frac{d\vec{r}}{dt}(t) \cdot v dt & \leq & \displaystyle\int_a^b \left\| \frac{d\vec{r}}{dt}(t) \right\| \cdot \|v\| dt \\ & = & \displaystyle \|v\| \int_a^b \left\| \frac{d\vec{r}}{dt}(t) \right\| dt \\ & = & \|v\| \cdot L(\Gamma) \end{array}$$

   Ocupando el resultado de a) se tiene que

$$\int_a^b \frac{d\vec{r}}{dt}(t) \cdot v dt = \|v\|^2 \leq \|v\| \cdot L(\Gamma)$$

   Tomando la ultima desigualdad y pasando diviendo $\|v\|$, se concluye que

$$\|v\| \leq L(\Gamma)$$

$\boxed{P3}$ a) Usemos $x = \cosh(t)$, $y = \operatorname{senh}(t)$

ya que reemplazando en $x^2 - y^2 = 1$ se tiene la identidad. Además

$$\tanh(z) = \frac{y}{x} = \frac{\operatorname{senh}(t)}{\cosh(t)} = \tanh(t) \Rightarrow z = t$$

Luego $\vec{r}(t) = (\cosh(t), \operatorname{senh}(t), t)$ con $t \in [0,1]$

pues $z \in [0,1]$

la longitud en un instante "t" es

$$S(t) = \int_0^t \left\| \frac{d\vec{r}(x)}{dx} \right\| dx = \int_0^t \|(\operatorname{senh}(x), \cosh(x), 1)\| dx$$

$$= \int_0^t \sqrt{\operatorname{senh}^2(x) + \cosh^2(x) + 1}\, dx$$

$$= \int_0^t \sqrt{2} \cosh(x)\, dx$$

$$= \sqrt{2} \operatorname{senh}(x) \Big|_0^t$$

$$= \sqrt{2} \operatorname{senh}(t)$$

$$\Rightarrow \quad L(\Gamma) = \int_0^1 \left\| \frac{d\vec{r}}{dt}(t) \right\| dt = S(1) = \sqrt{2} \operatorname{senh}(1)$$

b) Para la parametrización natural debemos despejar t en la relacion $s = s(t)$

$$\Rightarrow \quad s = s(t) = \sqrt{2}\,\operatorname{senh}(t)$$

$$\Rightarrow \quad t = \operatorname{senh}^{-1}\left(\frac{s}{\sqrt{2}}\right) \rightsquigarrow \text{reemplazo en } \vec{r}(t)$$

$$\Rightarrow \quad \vec{r}(s) = \vec{r}(t(s)) = \begin{pmatrix} \cosh(\operatorname{senh}^{-1}(\frac{s}{\sqrt{2}})) \\ \operatorname{Senh}(\operatorname{senh}^{-1}(\frac{s}{\sqrt{2}})) \\ \operatorname{Senh}^{-1}(\frac{s}{\sqrt{2}}) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{s^2}{2}+1} \\ s/\sqrt{2} \\ \operatorname{Senh}^{-1}(\frac{s}{\sqrt{2}}) \end{pmatrix}$$

c) Tangente $\vec{T}(t) = \dfrac{\frac{d\vec{r}(t)}{dt}}{\left\| \frac{d\vec{r}(t)}{dt} \right\|}$ ó $\vec{T}(s) = \dfrac{d\vec{r}(s)}{ds}$

$$\Rightarrow \quad \vec{T}(t) = \frac{(\operatorname{senh}(t), \cosh(t), 1)}{\sqrt{2}\cosh(t)}, \quad ó \quad \vec{T}(s) = \left(\frac{s}{\sqrt{2}\sqrt{s^2+2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{s^2+2}} \right)$$

Normal: $N(s) = \dfrac{\frac{dT(s)}{ds}}{\left\| \frac{dT(s)}{ds} \right\|} = \dfrac{\left(\frac{\sqrt{2}}{\sqrt{s^2+2}^3}, 0, \frac{s}{\sqrt{s^2+2}^3} \right)}{\frac{1}{s^2+2}}$

Binormal $\rightarrow$ ustedes

$\boxed{P4}$ tomando $\vec{P}_0 = \vec{r}(s) + \phi(s)\vec{N}(s)$ y derivando

$$\frac{d\vec{P}_0}{ds} = \frac{d}{ds}\left(\vec{r}(s) + \phi(s)\vec{N}(s)\right)$$

$$\Rightarrow \quad 0 = \frac{d\vec{r}(s)}{ds} + \frac{d(\phi \cdot \vec{N})(s)}{ds}$$ ← regla del producto

$$0 = \frac{d\vec{r}(s)}{ds} + \phi'(s)\vec{N}(s) + \phi(s)\frac{d\vec{N}}{ds}(s)$$

Frenet

$$0 = \vec{T}(s) + \phi'(s)\vec{N}(s) - K(s)\phi(s)\vec{T}(s) + \phi(s)\tau(s)\vec{B}(s)$$

$$0 = \underbrace{(1 - K(s)\phi(s))}_{\lambda_1}\vec{T}(s) + \underbrace{\phi'(s)}_{\lambda_2}\vec{N}(s) + \underbrace{\phi(s)\tau(s)}_{\lambda_3}\vec{B}(s)$$

Como $\vec{T}(t)$, $\vec{N}(t)$ y $\vec{B}(t)$ son L.i.

$$\Rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0$$

$$\Rightarrow \quad 1 - K(s)\phi(s) = 0 \Rightarrow K(s)\phi(s) = 1 \Rightarrow \phi(s) \neq 0$$

$$\phi'(s) = 0 \Rightarrow \phi(s) = cte$$

$$\phi(s)\tau(s) = 0 \Rightarrow \tau(s) = 0 \Rightarrow \text{curva plana}$$