AUXILIAR 14

P1 $f: [0, +\infty) \to \mathbb{R}$ creciente, continua y derivable en $(0, +\infty)$
tal que $f(0) = 0$

Definimos $F(x) = x \int_0^x f^2(t)\,dt$

· F es creciente

$$F'(x) = \int_0^x f^2(t)\,dt + x\left[f^2(x) - f^2(0)\cdot 0\right]$$

$$= \underbrace{\int_0^x f^2(t)\,dt}_{\geq 0} + \underbrace{x f^2(x)}_{\geq 0}$$

$\Rightarrow$ F es creciente en $[0, +\infty)$

· F es convexa

$$F''(x) = f^2(x) + f^2(x) + 2x\, f(x) \cdot f'(x).$$

$$= 2f^2(x) + 2x\, \underset{\substack{\downarrow \\ f(0)=0 \text{ y luego crece}}}{f(x)}\, \underbrace{f'(x)}_{>0} \geq 0$$

$\Rightarrow$ F es convexa en $[0, +\infty)$

P2

$$a_0 + a_1 + \dots + a_{n-1} = \frac{5n^2 - 3n + 2}{n^2 - 1}$$

$$a_0 + a_1 + \dots + a_{n-1} + a_n = \frac{5(n+1)^2 - 3(n+1) + 2}{(n+1)^2 - 1}$$

$$= \frac{5n^2 + 10n + 5 - 3n - 3 + 2}{n^2 + 2n + 1 - 1}$$

$$= \frac{5n^2 + 7n + 4}{n^2 + 2n}$$

restando $\Rightarrow$

$$a_n = \frac{5n^2 + 7n + 4}{n^2 + 2n} - \frac{5n^2 - 3n + 2}{n^2 - 1}$$

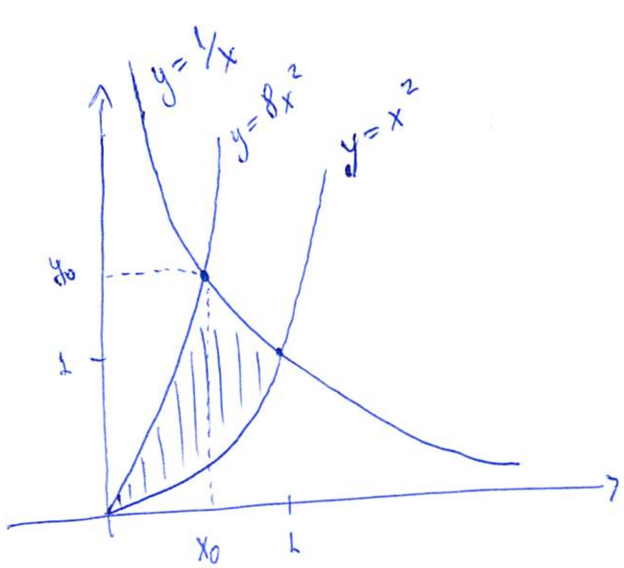
$$= \frac{\cancel{5n^4} + \cancel{7n^3} + 4n^2 - 5n^2 - 7n - 4 - \cancel{5n^4} + \cancel{3n^3} - 2n^2 - \cancel{10n^3} + 6n^2 - 4n}{(n^2 + 2n)(n^2 - 1)}$$

$$= \frac{3n^2 - 11n - 4}{(n^2 + 2n)(n^2 - 1)}$$

Además $\sum_{k=0}^{n-1} a_k = \frac{5n^2 - 3n + 2}{n^2 - 1} \xrightarrow{n \to \infty} \sum_{k=0}^{\infty} a_k = \lim_{n \to \infty} \frac{5n^2 - 3n + 2}{n^2 - 1}$

$$= 5$$

P3



Necesitamos x_0 para obtener el área.

$$\Rightarrow 8x_0^2 = \frac{1}{x_0} \Rightarrow x_0^3 = \frac{1}{8} \Rightarrow x_0 = \frac{1}{2}$$

$$A = \int_0^{1/2} 8x^2 - x^2\,dx + \int_{1/2}^{1} \frac{1}{x} - x^2\,dx$$

$$= 7\frac{x^3}{3}\Big|_0^{1/2} + \ln(x)\Big|_{1/2}^{1} - \frac{x^3}{3}\Big|_{1/2}^{1}$$

$$= \frac{7}{3}\cdot\frac{1}{8} + \ln(1) - \ln(1/2) + \frac{1}{3}\left[\frac{1}{8} - 1\right]$$

$$= \frac{7}{24} + \ln(2) - \frac{7}{24} = \ln(2).$$

P4 III)

$$r(t) = \frac{\sqrt{2}}{2}\begin{pmatrix} \text{sen}\, t \\ \cos t \\ t \end{pmatrix} \qquad t \in [0, 2\pi]$$

Calculemos $s(t) = \int_0^t \|r'\|$

$$\frac{dr}{dt} = \frac{\sqrt{2}}{2}\begin{pmatrix} \cos t \\ -\text{sen}\, t \\ 1 \end{pmatrix} \Rightarrow \|r'\| = \frac{\sqrt{2}}{2}\sqrt{\cos^2 t + \text{sen}^2 t + 1} = 1.$$

$$\Rightarrow s(t) = \int_0^t du = t \longrightarrow$$ Está parametrizado en longitud de arco

$$\vec{T}(t) = \frac{\sqrt{2}}{2}\begin{pmatrix} \cos t \\ -\text{sen}\, t \\ 1 \end{pmatrix} \underset{s(t)=t}{=} \frac{\sqrt{2}}{2}\begin{pmatrix} \cos s \\ -\text{sen}\, s \\ 1 \end{pmatrix}$$

$$T'(t) = \frac{\sqrt{2}}{2}\begin{pmatrix} -\text{sen}\, t \\ -\cos t \\ 0 \end{pmatrix} \qquad \|T'\| = \frac{\sqrt{2}}{2}\sqrt{\text{sen}^2 t + \cos^2 t} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow N(t) = \begin{pmatrix} -\text{sen}\, t \\ -\cos t \\ 0 \end{pmatrix} = \begin{pmatrix} -\text{sen}\, s \\ -\cos s \\ 0 \end{pmatrix}$$

Sigue que $\quad K(s) = \frac{\sqrt{2}}{2}$

Para calcular B, tenemos

$$B(s) = T(s) \times N(s)$$

$$= \frac{\sqrt{2}}{2}\begin{pmatrix} \cos s \\ -\operatorname{sen} s \\ 1 \end{pmatrix} \times \begin{pmatrix} -\operatorname{sen} s \\ -\cos s \\ 0 \end{pmatrix}$$

$$= \frac{\sqrt{2}}{2}\begin{vmatrix} i & j & k \\ \cos s & -\operatorname{sen} s & 1 \\ -\operatorname{sen} s & -\cos s & 0 \end{vmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \cos s \\ -\operatorname{sen} s \\ -1 \end{pmatrix}$$

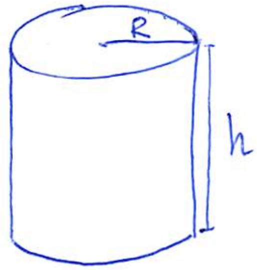
$$\Rightarrow B'(s) = \frac{\sqrt{2}}{2}\begin{pmatrix} -\operatorname{sen} s \\ -\cos s \\ 0 \end{pmatrix}$$

Con lo que

$$\tau(s) = -\begin{pmatrix} -\operatorname{sen} s \\ -\cos s \\ 0 \end{pmatrix} \cdot \frac{\sqrt{2}}{2}\begin{pmatrix} -\operatorname{sen} s \\ -\cos s \\ 0 \end{pmatrix} \cdot \frac{1}{1}.$$

$$= -\left[\operatorname{sen}^2 s + \cos^2 s\right]\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$$

P5



cumpliendo $R = f(h)$.

$$V(h) = \pi R^2 \cdot h = \pi h f^2(h).$$

Caso $f(x) = e^{-x}$

$$V(h) = \pi h e^{-2h}$$

$$V'(h) = \pi e^{-2h} - 2\pi h e^{-2h} = 0$$

$$2h = 1 \to h = 1/2.$$

$$V''(h) = -4\pi e^{-2h} + 4\pi h e^{-2h}$$

$$V''(1/2) = -4\pi + 4\pi \cdot \frac{1}{2} = -2\pi \to \text{es máximo}$$

$$\boxed{\begin{array}{l} h = 1/2 \\ R = e^{-1/2} \end{array}}$$

Si $f(x) = e^{-x^2} \Rightarrow V(h) = \pi h e^{-2^{2}}$

$$V'(h) = \pi e^{-2h^2} - 4h\,\pi h e^{-2h^2} = 0.$$

$$4h^2 = 1 \to h = 1/2$$

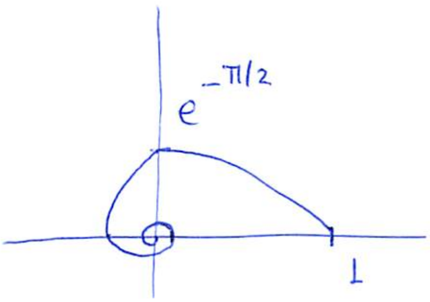
$$V''(h) = -12h\pi e^{-2h^2} + 16\pi h^3 e^{-2h^2}$$

$$V''(1/2) = e^{-1/2}\pi\left[-6 + 2\right] < 0. \to \text{es máximo}$$

$$\boxed{\begin{array}{l} h = 1/2 \\ R = e^{-1/4} \end{array}}$$

P6

$$\rho(\phi) = e^{-\phi} \qquad \phi \geq 0.$$



$$A(\phi) = \frac{1}{2}\int_0^\phi \rho^2(u)\,du$$

$$= \frac{1}{2}\int_0^\phi e^{-2u}\,du$$

Llamemos

$$A_n = \frac{1}{2}\int_{2(n-1)\pi}^{2n\pi} e^{-2\phi}\,d\phi$$

$$= \frac{1}{2}\left[-\frac{1}{2}e^{-2\phi}\right]_{2(n-1)\pi}^{2n\pi}$$

$\Rightarrow$ Área vuelta n: $A_n = -\frac{1}{4}\left[e^{-4\pi n} - e^{-4\pi(n-1)}\right]$

$$= -\frac{e^{-4\pi(n-1)}}{4}\left(e^{-4\pi} - 1\right)$$

Área vuelta 1: $A_1 = \frac{1}{4}\left(1 - e^{-4\pi}\right)$

$$\Rightarrow \quad A_n = e^{-4\pi(n-1)} A_1$$

$\boxed{P7}$

$a \in \mathbb{R}, \quad b > 0$

f par integrable en $[-a, a]$

$$\int_{-a}^{a} \frac{f(x)+1}{b^x+1}\,dx = I$$

$u = -x$
$du = -dx$

$$\Rightarrow I = \int_{a}^{-a} -\frac{f(-u)+1}{b^{-u}+1}\,du$$

$$= \int_{-a}^{a} \frac{b^u(f(u)+1)}{b^u+1}\,du$$

Sumando

$$\Rightarrow 2I = \int_{-a}^{a}\left[\frac{f(x)+1}{b^x+1} + b^x\frac{(f(x)+1)}{b^x+1}\right]dx$$

$$= \int_{-a}^{a} f(x)+1\,dx = 2a + 2\int_0^a f(x)\,dx$$

$$\Rightarrow I = a + \int_0^a f(x)\,dx.$$

Con lo que

$$\int_{-12345678910\pi}^{12345678910\pi} \frac{\cos^{123456789}(x)+1}{2016\,2017\,2018^x+1}\,dx = 12345678910\,\pi + \int_0^{12345678910\pi} \cos^{123456789}(x)\,dx$$

$$= 12345678910\,\pi.$$

$\boxed{P_3}$ $\rho(x,y,z) = 2x + 9z$, $\vec{r}(t) = (\underset{x}{t}, \underset{y}{t^2}, \underset{z}{t^3})$

$$\Rightarrow M = \int_{\Gamma} \rho \, d\ell = \int_0^3 \rho(\vec{r}(t)) \left\| \frac{d\vec{r}(t)}{dt} \right\| dt$$

$$= \int_0^3 2t + 9t^3 \, \| (1, 2t, 3t^2) \| \, dt$$

$$= \int_0^3 2t + 9t^3 \sqrt{1 + 4t^2 + 9t^4} \, dt$$

$$= \quad \ldots \quad \text{cambio } u = 1 + 4t^2 + 9t^4$$

$$= \frac{1}{6} \left((766)^{3/2} - 1 \right)$$

El centro de masa es

$$X_G = M \int_0^3 x \rho \, d\ell = M \int_0^3 t \rho \, d\ell = M \int_0^3 t \underbrace{(2t+9t^3)\sqrt{1+4t^2+9t^4}}_{\%} \, dt$$

$$Y_G = M \int_0^3 t^2 \, \% \, dt$$

$$Z_G = M \int_0^3 t^3 \, \% \, dt$$

Scanned by CamScanner

P2. [Fibonacci]

Recuerde la sucesión de Fibonacci, dada por $a_1 = a_2 = 1$ y $a_{n+1} = a_n + a_{n-1}$.

a) Definamos:

$$f(x) = \sum_{n=1}^{\infty} a_n x^{n-1} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots$$

Demuestre que la serie anterior converge cuando $|x| < 1/2$.
Obs*: Puede usar sin demostrar que* $\lim \frac{a_{n+1}}{a_n}$ *existe.*

b) Demuestre que si $|x| < 1/2$:

$$f(x) = \frac{-1}{x^2 + x - 1}$$

c) [**Propuesto**] Encuentre "otro" desarrollo en series de potencias para f.
Hint: *Parta factorizando* $x^2 + x - 1$.

d) [**Propuesto**] Concluya de las partes anteriores una forma cerrada para la sucesión de Fibonacci.
Hint: *Use la P1 d).*

Solución 2.

a) Utilicemos el criterio del cuociente:

$$\lim_n \frac{a_{n+1}|x|^{n+1}}{a_n|x^n|} = |x| \underbrace{\lim \frac{a_{n+1}}{a_n}}_{\text{existe por } \textbf{Obs.}} = |x| \lim \frac{a_n + a_{n-1}}{a_n} = |x| \lim \left(1 + \underbrace{\frac{a_{n-1}}{a_n}}_{\leq 1}\right) \leq 2|x|$$

Sabemos, entonces que si $|x| < 1/2$, entonces el límite anterior es menor a 1 y por tanto converge.

b) Notemos que:

$$x^2 f(x) = x^2 \sum_{k\geq 1} a_k x^{k-1} = \sum_{k\geq 1} a_k x^{k+1} = \sum_{k\geq 3} a_{k-2} x^{k-1}$$

$$x f(x) = x \sum_{k\geq 1} a_k x^{k-1} = \sum_{k\geq 1} a_k x^k = \sum_{k\geq 2} a_{k-1} x^{k-1} = a_1 x + \sum_{k\geq 3} a_{k-1} x^{k-1} = x + \sum_{k\geq 3} a_{k-1} x^{k-1}$$

$$f(x) = \sum_{k\geq 1} a_k x^{k-1} = a_1 + a_2 x + \sum_{k\geq 3} a_k x^{k-1} = 1 + x + \sum_{k\geq 3} a_k x^{k-1}$$

Por tanto:

$$x^2 f(x) + x f(x) - f(x) = -1 + \sum_{k\geq 3} (\underbrace{a_{k-2} + a_{k-1} - a_k}_{=0}) x^{k-1} = -1$$

Factorizando por $f(x)$ y dividiendo tenemos que:

$$f(x) = \frac{-1}{x^2 + x - 1}$$

c) Las raíces de $x^2 + x - 1$ son $\alpha = \frac{-1+\sqrt{5}}{2}$ y $\beta = \frac{-1-\sqrt{5}}{2}$. Por tanto:

$$\begin{aligned}
f(x) &= \frac{-1}{(\alpha - x)(\beta - x)} \\
&= \frac{-1}{\alpha\beta} \underbrace{\frac{1}{(1-\frac{x}{\alpha})}}_{\text{Geométrica}} \underbrace{\frac{1}{(1-\frac{x}{\beta})}}_{\text{Geométrica}} \\
&= \frac{-1}{\alpha\beta} \left(\sum_{k\geq 0} \frac{1}{\alpha^k} x^k\right) \left(\sum_{k\geq 0} \frac{1}{\beta^k} x^k\right) \\
&= \frac{-1}{\alpha\beta} \sum_{k\geq 0} \left(\sum_{j=0}^{k} \frac{1}{\alpha^j} \frac{1}{\beta^{k-j}}\right) x^k \\
&= \frac{-1}{\alpha\beta} \sum_{k\geq 0} \frac{1}{\beta^k} \left(\sum_{j=0}^{k} \left[\frac{\beta}{\alpha}\right]^j\right) x^k \\
&= \frac{-1}{\alpha\beta} \sum_{k\geq 0} \frac{1}{\beta^k} \frac{(\beta/\alpha)^{k+1} - 1}{\beta/\alpha - 1} x^k \\
&= \frac{-1}{\alpha\beta} \sum_{k\geq 0} \frac{1}{\beta^k} \frac{\frac{\beta^{k+1} - \alpha^{k+1}}{\alpha^{k+1}}}{\frac{\beta - \alpha}{\alpha}} x^k \\
&= \frac{-1}{\alpha\beta} \sum_{k\geq 0} \frac{1}{\alpha^k \beta^k} \frac{\beta^{k+1} - \alpha^{k+1}}{\beta - \alpha} x^k \\
&= -\sum_{k\geq 0} \frac{\frac{1}{\alpha^{k+1}} - \frac{1}{\beta^{k+1}}}{\beta - \alpha} x^k \\
&= \sum_{k\geq 0} \frac{\frac{1}{\beta^{k+1}} - \frac{1}{\alpha^{k+1}}}{\beta - \alpha} x^k \\
&= \sum_{k\geq 1} \frac{\frac{1}{\beta^{k}} - \frac{1}{\alpha^{k}}}{\beta - \alpha} x^{k-1}
\end{aligned}$$

Esto es "otra" expansión en serie de potencia de $f(x)$.

d) Por la P1 d) sabemos que si tenemos dos expansiones en series de potencias deben ser iguales término a término, de donde concluimos la siguiente forma cerrada (no recursiva) para Fibonacci:

$$\begin{aligned}
a_n &= \frac{\frac{1}{\beta^n} - \frac{1}{\alpha^n}}{\beta - \alpha} \\
&= \frac{\frac{1}{\beta^n} - \frac{1}{\alpha^n}}{\beta - \alpha} \\
&= \frac{\frac{1}{\beta^n} - \frac{1}{\alpha^n}}{-\sqrt{5}} \\
&= \frac{1}{\sqrt{5}} \left(\frac{1}{\alpha^n} - \frac{1}{\beta^n}\right)
\end{aligned}$$

Obs: Trabajando un poco esto se puede llegar a la Fórmula de Binet:

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$