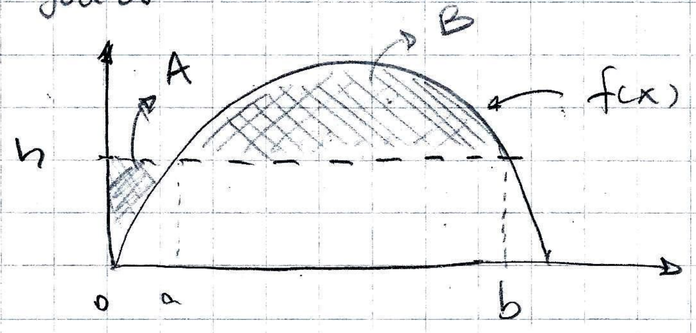
Auxilear #9

P1 $f(x) = 2x - 3x^3$, determinar h para que A y B sean iguales



$$A = \text{Area } \underset{0 \quad a}{\square}\, h - \text{Area } \underset{0 \quad a}{\frown}$$

$$= \int_0^a h\,dx - \int_0^a f(x)\,dx = \int_0^a (h - f(x))\,dx$$

$$B = \int_a^b (f(x) - h)\,dx$$

$$\Rightarrow \quad A = B \iff \int_0^a h - f(x)\,dx = \int_a^b f(x) - h$$

$$\Rightarrow \int_0^a h\,dx + \int_a^b h\,dx = \int_0^a f(x)\,dx + \int_a^b f(x)\,dx$$

$$= \int_0^b h\,dx = \int_0^b f(x)\,dx$$

$$= h(b-0) = \int_0^b 2x - 3x^3\,dx$$

$$\Rightarrow \quad bh = \int_0^b 2x - 3x^3\,dx = \left(x^2 - \frac{3}{4}x^4\right)\Big|_0^b$$

$$= b^2 - \frac{3}{4}b^4$$

$$\Rightarrow \quad bh = b^2 - \frac{3}{4}b^4 \Rightarrow h = b - \frac{3}{4}b^3 \qquad (1)$$

por otro lado $f(b) = h \Rightarrow h = f(b) = 2b - 3b^3 \qquad (2)$

juntando $b - \frac{3}{4}b^3 = 2b - 3b^3$

$$\Rightarrow \quad 0 = b - 3b^3 + \frac{3}{4}b^3$$

$$= b - \frac{9}{4}b^3$$

$$= b\left(1 - \frac{9}{4}b^2\right)$$

$$\Rightarrow \quad b = 0 \quad \vee \quad b^2 = \frac{4}{9} \Rightarrow b = \pm\frac{2}{3}$$

Nos quedamos con $b = \frac{2}{3}$ pues $b > 0$

$\Rightarrow$ evaluando en (1) o (2)

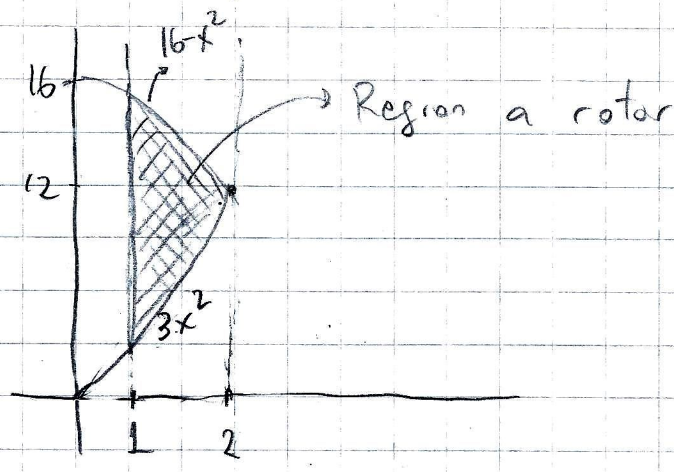
$$\Rightarrow \quad h = \frac{4}{9}\,u$$

$\searrow$ unidades

$\boxed{P2}$ hallar Volumen al rotar en torno al eje OY la región encerrado por

$$y_1 = 3x^2 \ , \ y_2 = 16 - x^2 \ , \ x = 1 \ \text{y} \ x = 2$$

Para ver que region encercamos debemos intersectar todas las curvas ya sea analiticamente o en el grafico



Notemos que $y_1 \cap y_2 \Rightarrow 3x^2 = 16 - x^2$

$$\Rightarrow 4x^2 = 16$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2,$$

nos quedamos con $x = 2$

Luego el volumen a integrar es el volumen que genera y_2 menos el volumen de y_1

$$\Rightarrow V = 2\pi \int_1^2 x y_2 \, dx - 2\pi \int_1^2 x y_1 \, dx = 2\pi \int_1^2 x (y_2 - y_1) \, dx$$

$$\Rightarrow V = 2\pi \int_1^2 x(16 - x^2 - 3x^2)\,dx$$

$$= 2\pi \int_1^2 x(16 - 4x^2)\,dx$$

$$= 2\pi \int_1^2 16x - 4x^3\,dx = 2\pi\left(8x^2 - x^4\right)\Big|_1^2$$

$$= 2\pi\left[8(2)^2 - (2)^4 - (8 - 1)\right]$$

$$= 2\pi\left[32 - 16 - 8 + 1\right]$$

$$= 18\pi \ u^3$$

unidades cubicas

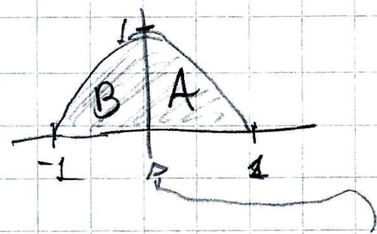
$\boxed{P3}$ a) $f(x) = \dfrac{1 - x^2}{1 + x^2}$, $f(x) \geq 0$

1) encontremos la region R.

Ceros de f: $f(x) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$

notar que $f(x) \geq 0 \quad \forall x \in [-1, 1]$

$\Rightarrow$ debemos estudiar lo sgte



$$\Rightarrow A = \int_{-1}^{1} f(x)\,dx$$ PERO! f es par $\Rightarrow A = B$

$$= 2\int_0^1 f(x)\,dx = 2\int_0^1 \frac{1 - x^2}{1 + x^2}\,dx$$

$$\Rightarrow A = 2\int_0^1 \frac{1-x^2}{1+x^2}\,dx = 2\int_0^1 \frac{1+1-1-x^2}{1+x^2}\,dx$$ NIKITA NEPONE

$$= 2\int_0^1 \frac{2-(1+x^2)}{1+x^2}\,dx$$

$$= 2\left(\int_0^1 \frac{2}{1+x^2} - \int_0^1 1\,dx\right)$$

$$= 2\left(2\arctan(x)\Big|_0^1 - x\Big|_0^1\right)$$

$$= 2\left(\left(2\arctan(x) - x\right)\Big|_0^1\right)$$

$$= 2\left(2\arctan(1) - 1 - \cancel{\left(2\arctan(0) - 0\right)}^{\,0}\right)$$

$$= 2\left(2\frac{\pi}{4} - 1\right)$$

$$= \pi - 2 \quad [u^2] \to \text{unidades cuadradas}$$

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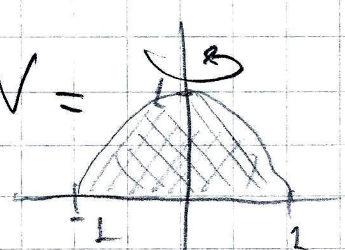
2) debemos rotar en torno a OY la sgte figura

V = 

pero para integrar necesitamos que

$0 \le a < b$

y aquí $a=-1$ y $b=1$

PERO! f es par, debido a esto calcular el volumen entre -1 y 1 es lo mismo que entre 0 y 1

ya que forman la misma figura

obs: no es necesario multiplicar por 2.

$$\Rightarrow V = 2\pi \int_0^1 x\, f(x)\, dx = 2\pi \int_0^1 x \frac{(1-x^2)}{1+x^2}\, dx$$

$$= 2\pi \int_0^1 x \left[\frac{2}{1+x^2} - 1\right] dx$$

$$= 2\pi \int_0^1 \frac{2x}{1+x^2} - x\, dx$$

$$= 2\pi \left[\underbrace{\int_0^1 \frac{2x}{1+x^2}\, dx}_{\text{Cambio de Variable}} - \int_0^1 x\, dx\right]$$

$u = x^2+1 \Rightarrow du = 2x\,dx$

$x=0 \Rightarrow u = 0^2+1 = 1$

$x=1 \Rightarrow u = 1^2+1 = 2$

$$= 2\pi \left[\int_1^2 \frac{du}{u} - \int_0^1 x\, dx\right]$$

$$= 2\pi \left[\ln(u)\Big|_1^2 - \frac{x^2}{2}\Big|_0^1\right]$$

$$= 2\pi \left[\ln(2) - \cancel{\ln(1)}^{\,0} - \left(\frac{1^2}{2} - \frac{0^2}{2}\right)\right]$$

$$= 2\pi \left[\ln(2) - \frac{1}{2}\right] \quad \boxtimes$$

b) $C_1 := x+y=5$, $C_2 := xy=4$.

Debemos intersectar las curvas para saber que región integrar

$\Rightarrow C_1 := y = 5-x$, $C_2 := y = \frac{4}{x}$

$\Rightarrow C_1 \cap C_2 \Rightarrow 5-x = \frac{4}{x} \quad / \cdot x$

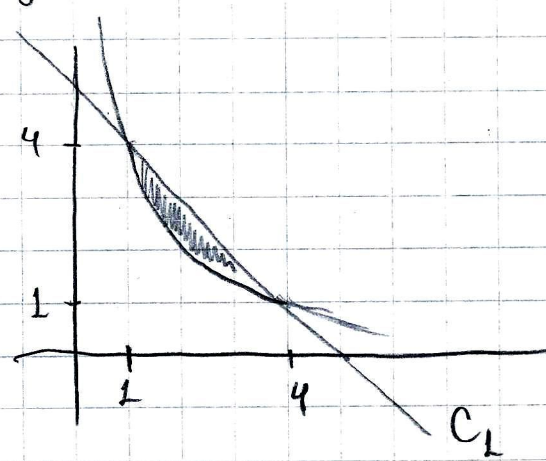
$\Rightarrow 5x - x^2 = 4$

$\Rightarrow 0 = x^2 - 5x + 4 = (x-4)(x-1)$

$\Rightarrow x = 1$ y $x = 4$

si $x=1 \Rightarrow y=4$
si $x=4 \Rightarrow y=1$

Luego la region es

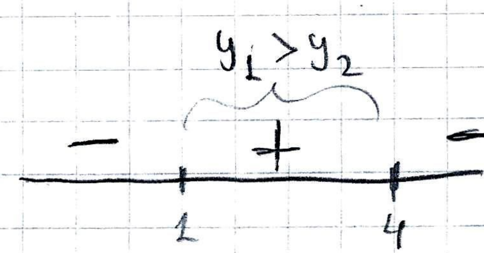


Notar que C_1 pasa por sobre C_2. Probemoslo

$y_1 > y_2 \Leftrightarrow 5-x > \frac{4}{x}$ Si $x>0$

$\Rightarrow 5x - x^2 > 4$

$\Rightarrow 5x - x^2 - 4 > 0$



Luego el volumen de la region es el volumen generado por el area bajo la recta menos el volumen generado por el area bajo C_2

En torno a OX

$$\Rightarrow V_{ox} = \pi \int_1^4 (5-x)^2 dx - \pi \int_1^4 \left(\frac{4}{x}\right)^2 dx$$

$$= \pi \left(\int_1^4 25 - 10x + x^2 - \frac{16}{x^2} \, dx \right)$$

$$= \pi \left[25x - 5x^2 + \frac{x^3}{3} + \frac{16}{x} \right]_1^4$$

$$= \pi \left[\cancel{100} - \cancel{80} + \frac{64}{3} + 4 - \left(\cancel{25} - \cancel{5} + \frac{1}{3} + 16 \right) \right]$$

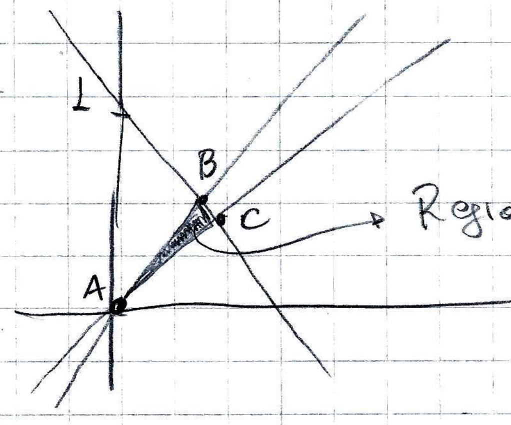
$$= \pi \left[\cancel{\frac{64}{3}} + 4 - \cancel{\frac{1}{3}} - 16 \right]$$

$$= \pi \left[21 + 4 - 16 \right]$$

$$= 9\pi \quad \blacksquare$$

$\boxed{P4}$ $y_1 = x$, $y_2 = \alpha x$, $y_3 = 1 - \alpha x$ $\alpha \geq 1$

Región encerrada



Debemos encontrar A, B y C para poder integrar

- $A \rightarrow y_1 \cap y_2 \Rightarrow x = \alpha x \Rightarrow x = 0 \Rightarrow y_1 = 0$

$A = (0,0)$

- $B \rightarrow y_2 \cap y_3 \Rightarrow \alpha x = 1 - \alpha x$ $\qquad B = \left(\frac{1}{2\alpha}, \frac{1}{2}\right)$

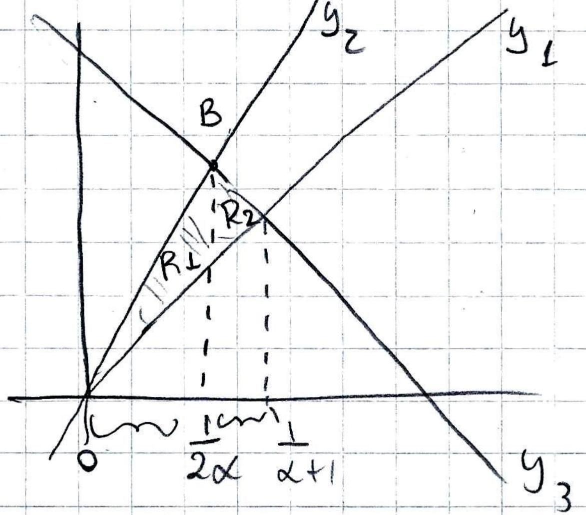
$$2\alpha x = 1$$
$$x = \frac{1}{2\alpha} \Rightarrow y_2 = \cancel{\alpha} \frac{1}{2\cancel{\alpha}} = \frac{1}{2}$$

- $C \rightarrow y_1 \cap y_3 \Rightarrow x = 1 - \alpha x$

$$\Rightarrow x(\alpha + 1) = 1$$
$$x = \frac{1}{\alpha + 1} \Rightarrow y_1 = \frac{1}{\alpha + 1}$$

$C = \left(\frac{1}{\alpha+1}, \frac{1}{\alpha+1}\right)$

notar que si hacemos zoom



Dos regiones

$$\Rightarrow A_R = \int_0^{\frac{1}{2\alpha}} y_2 - y_1 \, dx + \int_{\frac{1}{2\alpha}}^{\frac{1}{\alpha+1}} y_3 - y_1 \, dx$$

$$= \int_0^{\frac{1}{2\alpha}} \alpha x - x \, dx + \int_{1/2\alpha}^{1/\alpha+1} \underbrace{1 - \alpha x - x}_{1-(\alpha+1)x} \, dx$$

$$= \left[\frac{\alpha}{2} x^2 - \frac{x^2}{2} \right]_0^{\frac{1}{2\alpha}} + \left[x - \frac{(\alpha+1)}{2} x^2 \right]_{\frac{1}{2\alpha}}^{\frac{1}{\alpha+1}}$$

$$= \left[\frac{\alpha}{2} \left(\frac{1}{2\alpha}\right)^2 - \frac{1}{2} \left(\frac{1}{2\alpha}\right)^2 \right] + \left[\frac{1}{\alpha+1} - \frac{(\alpha+1)}{2} \left(\frac{1}{\alpha+1}\right)^2 - \left(\frac{1}{2\alpha} - \frac{(\alpha+1)}{2} \left(\frac{1}{2\alpha}\right)^2 \right.\right.$$

$$= \frac{\cancel{\alpha}}{2} \frac{1}{4\alpha^{\cancel{2}}} - \frac{1}{2} \frac{1}{4\alpha^2} + \left[\frac{1}{\alpha+1} - \frac{\cancel{\alpha+1}}{2} \cdot \frac{1}{(\alpha+1)^{\cancel{2}}} - \frac{1}{2\alpha} + \frac{\alpha+1}{2} \cdot \frac{1}{4\alpha^2} \right]$$

$$= \frac{1}{8\alpha} - \frac{1}{8\alpha^2} + \frac{1}{\alpha+1} - \frac{1}{2\alpha} - \frac{1}{2(\alpha+1)} + \frac{\alpha+1}{8\alpha^2}$$

$$\rightarrow A_R = \frac{1}{8\alpha} - \frac{1}{2\alpha} + \frac{\alpha+1}{8\alpha^2} - \frac{1}{8\alpha^2} + \frac{1}{\alpha+1} - \frac{1}{2(\alpha+1)}$$

$$= -\frac{3}{8\alpha} + \frac{\alpha}{8\alpha^2} + \frac{1}{2(\alpha+1)}$$

$$= \frac{1}{2(\alpha+1)} - \frac{1}{4\alpha}$$

Ahora el volumen, analogamente

$$V_R = \pi \int_0^{\frac{1}{2\alpha}} y_2^2 - y_1^2 \, dx + \pi \int_{\frac{1}{2\alpha}}^{\frac{1}{\alpha+1}} y_3^2 - y_1^2 \, dx$$

$$= \pi \int_0^{\frac{1}{2\alpha}} \alpha^2 x^2 - x^2 \, dx + \pi \int_{\frac{1}{2\alpha}}^{\frac{1}{\alpha+1}} (1-\alpha x)^2 - x^2 \, dx$$

$$= \pi \int_0^{\frac{1}{2\alpha}} (\alpha^2 - 1) x^2 \, dx + \pi \int_{\frac{1}{2\alpha}}^{1/\alpha+1} 1 - 2\alpha x + \alpha x^2 - x^2$$

$$= \pi(\alpha^2-1) \int_0^{\frac{1}{2\alpha}} x^2 \, dx + \pi \left(\int_{\frac{1}{2\alpha}}^{1/\alpha+1} 1 \, dx - 2\alpha \int_{1/2\alpha}^{1/\alpha+1} x \, dx + (\alpha-1) \int_{1/2\alpha}^{1/\alpha+1} x^2 \, dx \right.$$

matraca ...

$$= \frac{\pi(\alpha^2 + 2\alpha - 3)}{12\alpha(\alpha+1)^2}$$

b) $A(\alpha) = \frac{1}{2(\alpha+1)} - \frac{1}{4\alpha}$

$\Rightarrow$ $A'(\alpha) = -\frac{1}{2(\alpha+1)^2} + \frac{1}{4\alpha^2} = 0$

$\Rightarrow$ $\frac{1}{2(\alpha+1)^2} = \frac{1}{4\alpha^2}$

$\Rightarrow$ $4\alpha^2 = 2(\alpha+1)^2 = 2(\alpha^2+2\alpha+1)$

$\Rightarrow$ $2\alpha^2 = \alpha^2 + 2\alpha + 1$

$\Rightarrow$ $\alpha^2 - 2\alpha - 1 = 0$

$$\alpha = \frac{2 \pm \sqrt{4 - 4(1)\cdot(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2}$$

$$= 1 \pm \sqrt{2}$$

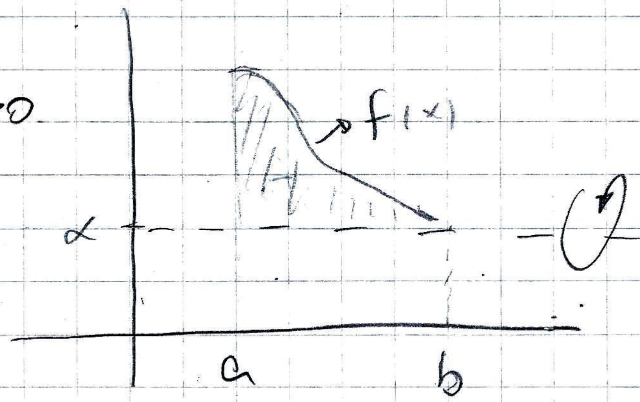
Como $\alpha \geq 1$ $\Rightarrow$ $\alpha = 1+\sqrt{2}$ es nuestro candidato

Notar que $A''(\alpha) = \frac{1}{(1+\alpha)^3} - \frac{1}{2\alpha^3}$

$\Rightarrow$ $A''(\sqrt{2}+1) = \frac{1}{(1+\sqrt{2}+1)^3} - \frac{1}{2(\sqrt{2}+1)^3} \geq 0$ $\therefore$

$\alpha = 1+\sqrt{2}$ es max local

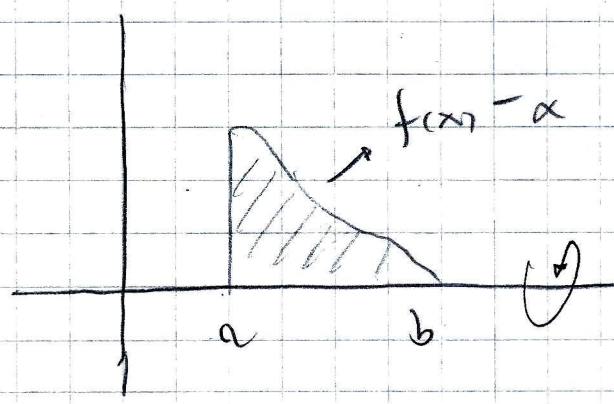
$\boxed{P5}$ a) Si $y = \alpha > 0$



$\downarrow \alpha$

Si este es el caso desplazamos todo hacia abajo

$\Rightarrow$



$\alpha = 0$

$$\Rightarrow V = \pi \int_a^b (f(x) - \alpha)^2 \, dx$$

Si $\alpha < 0$ es analogo $V = \pi \int_a^b (f(x) + \alpha)^2 \, dx$

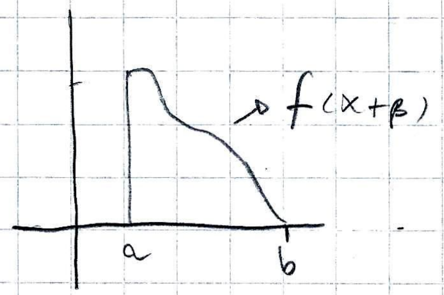
Si $X = \beta > 0$

$\xleftarrow{\beta}$

} conservar las distancias

Movemos todo a la izquierda, pero recuerden que si quiero mover a la izq una función debo sumar en el argumento! Ej: mover f 1 a la izq $\Rightarrow f(x+1)$

$\Rightarrow$ Cambiando el sistema queda



$$\Rightarrow 2\pi \int_{a-\beta}^{b-\beta} x f(x+\beta)\,dx$$ haciendo $u = x+\beta$ $du = dx$

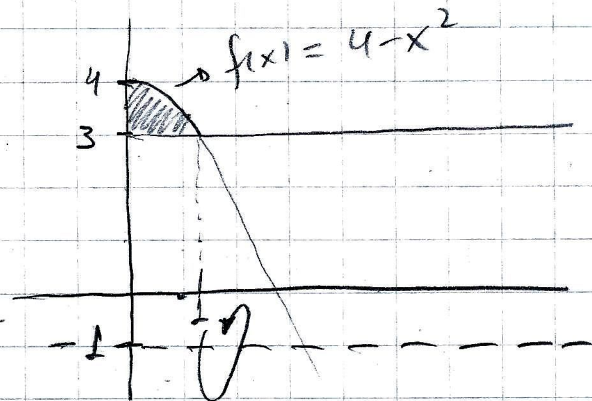
$x = b-\beta \Rightarrow u = b$
$x = a-\beta \Rightarrow u = a$

$$= 2\pi \int_a^b (u-\beta) f(u)\,du$$

$$\Rightarrow \quad V = 2\pi \int_a^b (x-\beta) f(x)\,dx$$

si $\beta < 0$ $\quad V = 2\pi \int_a^b (x+\beta) f(x)\,dx$

b) rotar en torno $y=-1$, de la región acotada por:
$y = 4-x^2$, $y=3$



$f(x)=3$
$\Rightarrow 4-x^2=3$
$\Rightarrow x^2=1 \Rightarrow x=\pm 1$

$\Rightarrow$ desplazamos en 1 hacia arriba y el volumen sería la parabola menos la recta

$$\Rightarrow \text{Volumen} = \pi \int_0^1 \left((4-x^2)+1\right)^2 - (3+1)^2\, dx$$

$$= \pi \int_0^1 (5-x^2)^2 - 4^2\, dx$$

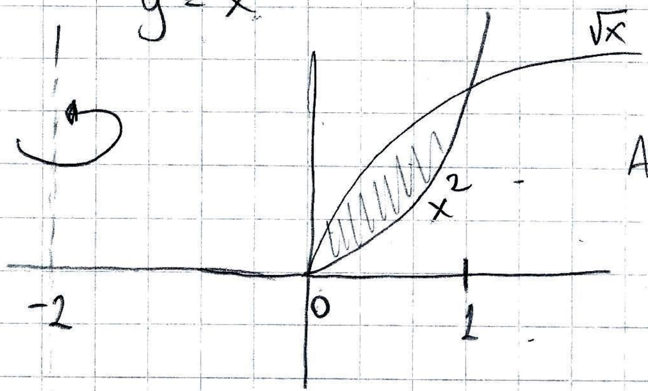
$$= \pi \int_0^1 25 - 10x^2 + x^4 - 16\, dx$$

$$= \pi \int_0^1 9 - 10x^2 + x^4\, dx$$

$$= \pi \left(9x - \frac{10}{3}x^3 + \frac{x^5}{5} \right)\Big|_0^1 = \pi \left(9 - \frac{10}{3} + \frac{1}{5} \right)$$

$$= \pi \left(\frac{176}{5} \right)$$

c) rotar en $x = -2$ la region encerrada por $y = \sqrt{x}$, $y = x^2$



Area de revolucion es sobre $\sqrt{x} - x^2$

desplazando hacia la derecha

$$\Rightarrow \quad \text{Volumen} = 2\pi \int_0^1 (x+2)(\sqrt{x} - x^2)\,dx$$

$$= 2\pi \int_0^1 x\sqrt{x} - x^3 + 2\sqrt{x} - 2x^2\,dx$$

$$= 2\pi \int_0^1 x^{3/2} - x^3 + 2x^{1/2} - 2x^2\,dx$$

$$= 2\pi \left[\frac{x^{5/2}}{5/2} - \frac{x^4}{4} + \frac{2x^{3/2}}{3/2} - \frac{2}{3}x^3 \right]_0^1$$

$$= 2\pi \left[\frac{2}{5} - \frac{1}{4} + \frac{4}{3} - \frac{2}{3} \right]$$

$$= 2\pi \left[\frac{2}{5} - \frac{1}{4} + \frac{2}{3} \right]$$

$$= 2\pi \left[\frac{24 - 15 + 40}{60} \right] = 2\pi \cdot \frac{49}{60} = \frac{49}{30}\pi$$