# Pauta Aux 9

**P1**

(a) $a(a^\dagger)^n\phi_0 = aa^\dagger(a^\dagger)^{n-1}\phi_0 = a^\dagger a(a^\dagger)^{n-1}\phi_0 + (a^\dagger)^{n-1}\phi_0 = a^\dagger a a^\dagger (a^\dagger)^{n-2}\phi_0 + (a^\dagger)^{n-1}\phi_0$

$= (a^\dagger)^2 a (a^\dagger)^{n-2}\phi_0 + 2(a^\dagger)^{n-1}\phi_0 = \ldots = n(a^\dagger)^{n-1}\phi_0$

(b) $a(Ce^{\alpha a^\dagger}\phi_0) = C\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}a(a^\dagger)^n\phi_0 = C\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}n(a^\dagger)^{n-1}\phi_0 = \alpha C e^{\alpha a^\dagger}\phi_0 = \alpha\varphi_\alpha$

$\langle\varphi_\alpha|\varphi_\alpha\rangle = 1 = |C|^2 \langle\phi_0|e^{\alpha^* a}e^{\alpha a^\dagger}|\phi_0\rangle = |C|^2 \langle\phi_0|e^{\alpha a^\dagger}e^{\alpha^* a}e^{[\alpha^* a,\alpha a^\dagger]}|\phi_0\rangle$

$= |C|^2 e^{|\alpha|^2} \Rightarrow \underline{|C| = 1/\sqrt{\exp(|\alpha|^2)}}$

(c) $(a^\dagger)\phi_n = \sqrt{n+1}\,\phi_{n+1}$, $(a^\dagger)^n\phi_0 = \sqrt{n!}\,\phi_n \Rightarrow \varphi_\alpha = C\sum\frac{\alpha^n}{\sqrt{n!}}\phi_n$

$\Rightarrow P_n = |C|^2|\alpha|^{2n}\cdot\frac{1}{n!} = \boxed{\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}}$

(d) $N = a^\dagger a \qquad \langle N\rangle = (\langle\varphi_\alpha|a^\dagger)(a|\varphi_\alpha\rangle) = |\alpha|^2$

(e) $H = \hbar\omega(N+\frac{1}{2})$. La acción de $H$ sobre sus autoestados la conocemos: $H\phi_n = \hbar\omega(n+\frac{1}{2})\phi_n$. Mientras tanto, $a\phi_n = \sqrt{n}\,\phi_{n-1}$ para $n\geq 1$. En estos casos claramente los autoestados de $H$ __no__ son estados coherentes. Pero __$\phi_0$__ sí lo es, con autovalor $\alpha = 0$.

(f) $\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2$. $\quad x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)$, $\quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)$

$\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2$.

$\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle\varphi_\alpha|(a+a^\dagger)^2|\varphi_\alpha\rangle = \frac{\hbar}{2m\omega}\left[\langle\varphi_\alpha|a^2|\varphi_\alpha\rangle + \langle\varphi_\alpha|(a^\dagger)^2|\varphi_\alpha\rangle + \underbrace{\langle\varphi_\alpha|aa^\dagger|\varphi_\alpha\rangle}_{\langle\varphi_\alpha|a^\dagger a|\varphi_\alpha\rangle + 1} + \langle\varphi_\alpha|a^\dagger a|\varphi_\alpha\rangle\right]$

$\Rightarrow \langle x^2\rangle = \frac{\hbar}{2m\omega}\left[\alpha^2 + (\alpha^*)^2 + 2|\alpha|^2 + 1\right]$. $\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\varphi_\alpha|a+a^\dagger|\varphi_\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\alpha+\alpha^*\right]$

$\Rightarrow \Delta x^2 = \frac{\hbar}{2m\omega}$

$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\left[\alpha^2 + (\alpha^*)^2 - \langle\varphi_\alpha|a^\dagger a|\varphi_\alpha\rangle - \langle\varphi_\alpha|aa^\dagger|\varphi_\alpha\rangle\right] = -\frac{\hbar m\omega}{2}\left[\alpha^2 + (\alpha^*)^2 - 2|\alpha|^2 - 1\right]$

$\langle p\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left[\alpha^* - \alpha\right] \Rightarrow \Delta p^2 = \frac{\hbar m\omega}{2}\{1\} \Rightarrow \Delta p\Delta x = \frac{\hbar}{2}$, incerteza mínima.

**P2** $\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\cdot\frac{1}{\sqrt{2\omega_k}}\left[e^{-ik_\mu x^\mu}a_k + e^{ik_\mu x^\mu}a_k^\dagger\right]$

$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\cdot\frac{1}{\sqrt{2\omega_k}}\cdot\frac{1}{\sqrt{2\omega_p}}\langle 0|\left(e^{-ik_\mu x_1^\mu}a_k + e^{ik_\mu x_1^\mu}a_k^\dagger\right)\left(e^{-ip_\mu x_2^\mu}a_p+e^{ip_\mu x_2^\mu}a_p^\dagger\right)|0\rangle$$

Pero $a_k|0\rangle = a_p|0\rangle = 0$, $\langle 0|a_p^\dagger = \langle 0|a_k^\dagger = 0$. Además, $\langle 0|a_k a_p^\dagger|0\rangle = \cancelto{0}{\langle 0|a_p^\dagger a_k|0\rangle} + \langle[a_k,a_p^\dagger$

$[a_k,a_p^\dagger] = (2\pi)^3\delta^3(\vec{k}-\vec{p}) \Rightarrow \langle\phi(x_1)\phi(x_2)\rangle = \int\frac{d^3k}{(2\pi)^3}\cdot\frac{1}{2\omega_k}e^{-ik_\mu(x_1^\mu - x_2^\mu)}$. Entonces,

$$\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = \langle 0|\phi(x_1)\phi(x_2)|0\rangle\theta(t_1-t_2) + \langle 0|\phi(x_2)\phi(x_1)|0\rangle\theta(t_2-t_1)$$

$$= \int\frac{d^3k}{(2\pi)^3}\left[e^{ik_\mu(x_2^\mu-x_1^\mu)}\theta(t_1-t_2) + e^{ik_\mu(x_1^\mu-x_2^\mu)}\theta(t_2-t_1)\right]\cdot\frac{1}{2\omega_k}$$

$$= \int \frac{d^3k}{(2\pi)^3}\left[e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}e^{-i\omega_k\tau}\theta(\tau) + e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}e^{i\omega_k\tau}\theta(-\tau)\right]\cdot\frac{1}{2\omega_k},\quad \tau\equiv t_1-t_2.$$

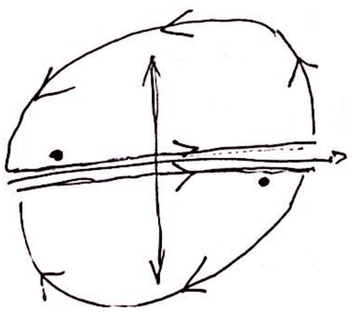
$= \int\frac{d^3k}{(2\pi)^3}\left[\frac{e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{2\omega_k}\right]\left[e^{-i\omega_k\tau}\theta(\tau)+e^{i\omega_k\tau}\theta(-\tau)\right]$. Usaremos la siguiente identidad:

$e^{-i\omega_k\tau}\theta(\tau)+e^{i\omega_k\tau}\theta(-\tau) = \lim_{\epsilon\to0}\frac{-2\omega_k}{2\pi i}\int\frac{d\omega}{\omega^2-\omega_k^2+i\epsilon}e^{i\omega\tau}$. Lo mostramos después.

$$\Rightarrow \langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = \lim_{\epsilon\to0}\int\frac{d^3k}{(2\pi)^3}\cdot\frac{e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{2\omega_k}\int d\omega\,\frac{-2\omega_k}{2\pi i}\cdot\frac{e^{i\omega\tau}}{\omega^2-\omega_k^2+i\epsilon}$$

$=\boxed{\lim_{\epsilon\to0}\int\frac{d^4k}{(2\pi)^4}\cdot\frac{i}{k_\mu k^\mu - m^2+i\epsilon}e^{ik_\mu(x_1^\mu-x_2^\mu)}}$ Donde ya no se cumple que $k_0^2 = \vec{k}^2+m^2$.

Dem. de la identidad: A $O(\epsilon)$, $\omega^2-\omega_k^2+i\epsilon = [\omega-(\omega_k-i\epsilon)][\omega-(-\omega_k+i\epsilon)]$, haciendo el cambio $\epsilon\leftrightarrow 2\epsilon\omega_k \Rightarrow \frac{1}{\omega^2-\omega_k^2+i\epsilon} \underset{\epsilon\to0}{=} \frac{1}{2\omega_k}\left[\frac{1}{\omega-(\omega_k-i\epsilon)} - \frac{1}{\omega+(\omega_k-i\epsilon)}\right]$ Dos polos simples



$I = \int\frac{e^{i\omega\tau}}{\omega^2-\omega_k^2+i\epsilon}d\omega$. Para $\tau>0$, cerramos el contorno por arriba. Al revés para $\tau<0$.

$$\int\frac{e^{i\omega\tau}d\omega}{\omega^2-\omega_k^2+i\epsilon} = \left[\underbrace{\int\frac{e^{i\omega\tau}d\omega}{\omega-(\omega_k-i\epsilon)}}_{-2\pi i e^{i\omega_k\tau}\theta(-\tau)} - \underbrace{\int\frac{e^{i\omega\tau}}{\omega+(\omega_k-i\epsilon)}}_{2\pi i e^{-i\omega_k\tau}\theta(\tau)}\right]\cdot\frac{1}{2\omega_k}$$

$$\Rightarrow \lim_{\epsilon\to0}\int\frac{d\omega}{\omega^2-\omega_k^2+i\epsilon}e^{i\omega\tau} = -\frac{2\pi i}{2\omega_k}\left[e^{i\omega_k\tau}\theta(-\tau)+e^{-i\omega_k\tau}\theta(\tau)\right]$$