Aux 4

$\boxed{P2}$ (a) $\frac{1}{4\pi\varepsilon_0} \approx 9\times 10^{9}\, Nm^2/C^2 \approx 10^{10}\, Nm^2/C^2$. $\quad r \sim a \Rightarrow r^3 \approx (5\times 10^{-11} m)^3 \approx 10^{2}\times 10^{-33} m^3 = 10^{-31} m^3$

$mc^2 \approx (10^{-30} kg)(10^{17} m^2/s^2) = 10^{-13} J \qquad L \sim \hbar \approx 10^{-34} Js$

$$\Rightarrow B \approx 10^{10}\, Nm^2/C^2 \cdot \frac{10^{-19} C}{10^{-13} J \cdot 10^{-31} m^3} \cdot 10^{-34} Js = 10T$$

$\Rightarrow$ Zeeman fuerte es $B \gg 10T$, pero esto es poco realista (record $B \approx 100T$).
Terrestre es $\approx 10^{-5}$, súper débil.

(b) Zeeman débil $\Rightarrow$ perturbamos sobre estructura fina. Queremos calcular $\langle H_Z \rangle$, con H_Z la perturbación con Zeeman. Este braket tiene autoestados de $H_0 + H_{FS}$, y este Hamiltoniano se diagonaliza con la base $|n\, l\, j\, m_j\rangle$ por culpa del acoplamiento spin-órbita, i.e. usamos autoestados de $\vec{J}^2$ y J_z, con $\vec{J} = \vec{L} + \vec{S}$.

(c)

$$|n,0,\tfrac{1}{2},\tfrac{1}{2}\rangle \underset{NB}{=} |n,0,0,\tfrac{1}{2}\rangle_{\otimes}$$

$$|n,0,\tfrac{1}{2},-\tfrac{1}{2}\rangle \underset{NB}{=} |n,0,0,-\tfrac{1}{2}\rangle_{\otimes}$$

$$\left.\begin{aligned} |2,1,\tfrac{1}{2},\tfrac{1}{2}\rangle &= -\sqrt{\tfrac{2}{3}}\,|2,1,1,-\tfrac{1}{2}\rangle_{\otimes} + \sqrt{\tfrac{1}{3}}\,|2,1,0,\tfrac{1}{2}\rangle_{\otimes} \\ |2,1,\tfrac{1}{2},-\tfrac{1}{2}\rangle &= -\sqrt{\tfrac{1}{3}}\,|2,1,0,-\tfrac{1}{2}\rangle_{\otimes} + \sqrt{\tfrac{2}{3}}\,|2,1,-1,\tfrac{1}{2}\rangle_{\otimes} \end{aligned}\right\} \begin{aligned} n&=2 \\ j&=\tfrac{1}{2} \\ l&=1 \end{aligned}$$

$$\left.\begin{aligned} |2,1,\tfrac{3}{2},\tfrac{3}{2}\rangle &= |2,1,1,\tfrac{1}{2}\rangle_{\otimes} \\ |2,1,\tfrac{3}{2},\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}\,|2,1,0,\tfrac{1}{2}\rangle_{\otimes} + \sqrt{\tfrac{1}{3}}\,|2,1,1,-\tfrac{1}{2}\rangle_{\otimes} \\ |2,1,\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \sqrt{\tfrac{1}{3}}\,|2,1,-1,\tfrac{1}{2}\rangle_{\otimes} + \sqrt{\tfrac{2}{3}}\,|2,1,0,-\tfrac{1}{2}\rangle_{\otimes} \\ |2,1,\tfrac{3}{2},-\tfrac{3}{2}\rangle &= |2,1,-1,-\tfrac{1}{2}\rangle_{\otimes} \end{aligned}\right\} \begin{aligned} n&=2 \\ j&=3/2 \\ l&=1 \end{aligned}$$

Para $n=1$, $\langle 1,0,\tfrac{1}{2},m_j | H_Z | 1,0,\tfrac{1}{2},m_j' \rangle$ es diagonal pues

$$H_Z = \frac{eB}{2m}(L_z+2S_z) \equiv \omega(L_z+2S_z) \Rightarrow \boxed{E_1^1 = \pm\hbar\omega}$$

Para $n=2$:

$$H_Z|2,1,\tfrac{3}{2},\pm\tfrac{3}{2}\rangle = \pm 2\omega\hbar\,|2,1,\tfrac{3}{2},\pm\tfrac{3}{2}\rangle$$

$$H_Z|2,0,\tfrac{1}{2},\pm\tfrac{1}{2}\rangle = \pm\hbar\omega\,|2,0,\tfrac{1}{2},\pm 1/2\rangle \qquad \boxed{E^1_{2,0,\frac{1}{2}} = \pm\hbar\omega}$$

$$\left.\begin{aligned} H_Z|2,1,\tfrac{1}{2},\tfrac{1}{2}\rangle &= \sqrt{\tfrac{1}{3}}\,\hbar\omega\,|2,1,0,\tfrac{1}{2}\rangle_{\otimes} \\ H_Z|2,1,\tfrac{1}{2},-\tfrac{1}{2}\rangle &= \sqrt{\tfrac{1}{3}}\,\hbar\omega\,|2,1,0,-\tfrac{1}{2}\rangle_{\otimes} \end{aligned}\right\} \quad \boxed{E^1_{2,1,\frac{1}{2}} = \pm\tfrac{1}{3}\hbar\omega}$$

$$H_Z|2,1,\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,\hbar\omega\,|2,1,0,\tfrac{1}{2}\rangle_{\otimes}$$

$$H_Z|2,1,\tfrac{3}{2},-\tfrac{1}{2}\rangle = -\sqrt{\tfrac{2}{3}}\,\hbar\omega\,|2,1,0,-\tfrac{1}{2}\rangle_{\otimes}$$

$\Rightarrow$ La matriz para $n=2$, $l=1$, $j=\tfrac{3}{2}$ también es diagonal, con

$$\boxed{E^1_{2,1,3/2} = \begin{matrix} \pm 2\hbar\omega \\ \pm\tfrac{2}{3}\hbar\omega \end{matrix}}$$

P3 $\frac{\partial}{\partial\lambda}\langle\psi_n|H|\psi_n\rangle = \frac{\partial}{\partial\lambda}E_n\langle\psi_n|\psi_n\rangle = \frac{\partial E_n}{\partial\lambda}$

$$= \left(\frac{\partial}{\partial\lambda}\langle\psi_n|\right)H|\psi_n\rangle + \langle\psi_n|\frac{\partial H}{\partial\lambda}|\psi_n\rangle + \langle\psi_n|H\left(\frac{\partial}{\partial\lambda}|\psi_n\rangle\right)$$

$$= \left\langle\frac{\partial H}{\partial\lambda}\right\rangle + E_n\cancel{\frac{\partial}{\partial\lambda}\langle\psi_n|\psi_n\rangle}^{\,0} = \left\langle\frac{\partial H}{\partial\lambda}\right\rangle$$

También podemos deducir esto con teoría de perturbaciones:

Si tenemos $H(\lambda)$, al hacer $\lambda_0 \to \lambda_0 + d\lambda$ tendremos $H = H(\lambda_0) + \underset{\text{Pert.}}{H'}$, con $H' = (\partial H/\partial\lambda)|_{\lambda_0}d\lambda$. $dE = \langle H'\rangle|_{\lambda_0} = \langle\psi_n|\frac{\partial H}{\partial\lambda}|\psi_n\rangle d\lambda|_{\lambda_0} \Rightarrow \frac{\partial E}{\partial\lambda} = \left\langle\frac{\partial H}{\partial\lambda}\right\rangle$.

P4 Aplicamos el teorema recién demostrado.

Tenemos $H = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$. Si buscamos $\langle\frac{1}{r}\rangle$, parece ser conveniente usar e como parámetro. $\partial H/\partial e = -\frac{e}{2\pi\varepsilon_0}\cdot\frac{1}{r}$

$$E = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2(j_{max}+l+1)^2} \qquad \frac{\partial E}{\partial e} = -\frac{me^3}{8\pi^2\varepsilon_0^2\hbar^2n^2} \Rightarrow \left\langle\frac{1}{r}\right\rangle = \frac{2\pi\varepsilon_0}{e}\cdot\frac{me^3}{8\pi^2\varepsilon_0^2\hbar^2n^2}$$

$$\Rightarrow \boxed{\left\langle\frac{1}{r}\right\rangle = \frac{me^2}{4\pi\varepsilon_0\hbar^2n^2}} \qquad \text{Además, } a = \frac{4\pi\varepsilon_0\hbar^2}{me^2} \Rightarrow \boxed{\left\langle\frac{1}{r}\right\rangle = \frac{1}{an^2}}$$

Para $\langle\frac{1}{r^2}\rangle$, nos va a servir l como parámetro.

$$\frac{\partial H}{\partial l} = \frac{\hbar^2}{2mr^2}\cdot(2l+1) \qquad \frac{\partial E}{\partial l} = \frac{me^4}{16\pi^2\varepsilon_0^2\hbar^2(j_{max}+l+1)^3}$$

$$\Rightarrow \boxed{\left\langle\frac{1}{r^2}\right\rangle = \frac{2m}{(2l+1)\hbar^2}\cdot\frac{me^4}{16\pi^2\varepsilon_0^2\hbar^2n^3}} \qquad a^2 = \frac{16\pi^2\varepsilon_0^2\hbar^4}{m^2e^4}$$

$$\Rightarrow \left\langle\frac{1}{r^2}\right\rangle = \frac{2}{(2l+1)a^2n^3} \qquad \boxed{\left\langle\frac{1}{r^2}\right\rangle = \frac{1}{(l+\frac{1}{2})a^2n^3}}$$