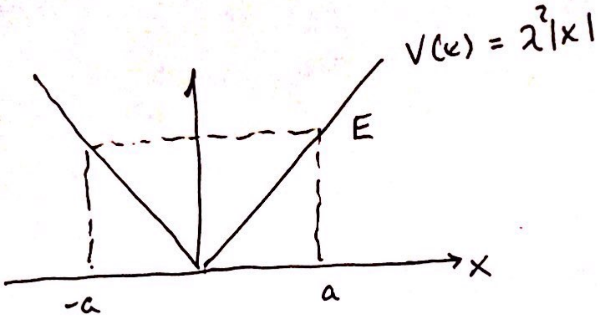
# Aux 3

Duda: hago resúmenes al principio?

**P1**




(a) únicos parámetros relevantes aquí son $\lambda, \hbar, m$

$V$ tiene dim. de $ML^2T^{-2} \Rightarrow \lambda^2$ tiene dim de $MLT^{-2}$. $\hbar$ tiene dim de $ML^2T^{-1}$. $E = \lambda^\alpha \hbar^\beta m^\gamma \to M^{\alpha/2+\beta+\gamma} L^{\alpha/2+2\beta} T^{-\alpha-\beta}$, $[E] = ML^2T^{-2}$

$$\Rightarrow \left.\begin{array}{ll} \frac{1}{2}\alpha+\beta+\gamma & = 1 \\ \frac{1}{2}\alpha + 2\beta & = 2 \\ -\alpha-\beta & = -2 \end{array}\right\} \quad 3\beta = 2 \Rightarrow \beta = 2/3 \Rightarrow \alpha = 4/3 \qquad \gamma = 1 - \frac{2}{3} - \frac{2}{3} = -1/3$$

$$\Rightarrow E \propto \lambda^{4/3} \cdot \hbar^{2/3} \cdot m^{-1/3}$$

(b) $$\psi(x) = \begin{cases} C\kappa^{-1/2} \exp\left[\int_x^a dx\, \kappa(x)\right] & x >> a \\ C' k^{1/2} \cos\left[\int_x^a dx\, k(x) - \frac{\pi}{4}\right] & x << a \end{cases} \qquad \begin{array}{l} k^2 = 2m(E-V)/\hbar^2 \\ \kappa^2 = 2m(V-E)/\hbar^2 \end{array}$$

Veamos $x > a$. Aquí, $H\psi = E\psi$, con $H = \frac{p^2}{2m} + \lambda^2 x = \frac{-\hbar^2}{2m}\partial_x^2 + \lambda^2 x$

$$\partial_x \exp\left[\int_x^a dx\, \kappa(x)\right] = \exp\left[\int_x^a dx\, \kappa(x)\right] \cdot -\kappa(x) \equiv A_1$$

$$\partial_x^2 \exp[\ ] = \exp\left[\int_x^a dx\, \kappa(x)\right]\left[+\kappa^2 - \kappa'\right] \equiv A_2$$

$$\Rightarrow \partial_x\psi = C\kappa^{-1/2}A_1 - \frac{1}{2}C\kappa^{-3/2}\kappa' \exp\left[\int_x^a dx\, \kappa(x)\right]$$

$$\partial_x^2\psi = -\frac{1}{2}C\kappa^{-3/2}\kappa' A_1 + C\kappa^{-1/2}A_2 + \frac{3}{4}C\kappa^{-5/2}(\kappa')^2 \exp\left[\int_x^a dx\, \kappa(x)\right]$$
$$- \frac{1}{2}C\kappa^{-3/2}\kappa'' \exp\left[\int_x^a dx\, \kappa(x)\right] + \frac{1}{2}C\kappa^{-3/2}\kappa' \exp\left[\int_x^a dx\, \kappa(x)\right]\kappa$$

Queremos $-\partial_x^2\psi = (E-V)\cdot 2m/\hbar^2\psi \Rightarrow \partial_x^2\psi = \kappa^2\psi = \kappa^2 C\kappa^{-1/2}\exp\left[\int_x^a dx\, \kappa(x)\right]$

Se cancelan los $C$ y las exp

$$\Rightarrow -\frac{1}{2}\kappa^{-3/2}\kappa'(-\kappa) + \kappa^{-1/2}\left[\cancel{\kappa^2} - \kappa'\right] + \frac{3}{4}\kappa^{-5/2}(\kappa')^2 - \frac{1}{2}\kappa^{-3/2}\kappa'' + \frac{1}{2}\kappa^{-3/2}\kappa'\kappa - \cancel{\kappa^{3/2}} = 0$$

$$\cancel{+\frac{1}{2}\kappa^{-1/2}\kappa'} \cancel{-\kappa^{-1/2}\kappa'} + \frac{3}{4}\kappa^{-5/2}(\kappa')^2 - \frac{1}{2}\kappa^{-3/2}\kappa'' \cancel{+\frac{1}{2}\kappa^{-1/2}\kappa'} = 0$$

$\Rightarrow$ Necesitamos $\frac{3}{4}\kappa^{-5/2}(\kappa')^2 - \frac{1}{2}\kappa^{-3/2}\kappa'' \approx 0$ ) se satisface con esto

$$\kappa' << \kappa^2 \Leftrightarrow \kappa'' << \kappa\kappa'$$

Porque es $\sim \frac{3}{4}\kappa^{-2}\kappa' - \frac{1}{2}\kappa^{-1}\frac{\kappa''}{\kappa'} \approx 0$ ✓

$$\kappa = \sqrt{\frac{2m(V-E)}{\hbar^2}} = \sqrt{\frac{2m(\lambda^2 x - E)}{\hbar^2}} \qquad \kappa' = \sqrt{\frac{2m}{\hbar^2}} \cdot \lambda^2 \cdot (V-E)^{-1/2}$$

$$\kappa' \ll \kappa^2 \Rightarrow \sqrt{\frac{2m}{\hbar^2}}\, \lambda^2 (V-E)^{-1/2} \ll \frac{2m}{\hbar^2}(V-E) \Rightarrow (V-E)^{3/2} \gg \frac{\lambda^2}{2}\sqrt{\frac{\hbar^2}{2m}}$$

Pero buscamos la distancia $L$ al punto $a$, i.e. $L = x - a$.

$$V(a) = E \Rightarrow V(x) - E = V(x) - V(a) = \lambda^2 (x-a) = \lambda^2 L$$

$$\Rightarrow \lambda^3 L^{3/2} \gg \frac{\lambda^2}{2}\sqrt{\frac{\hbar^2}{2m}} \Rightarrow \boxed{L \gg \left(\frac{\hbar^2}{M\lambda^2}\right)^{1/3}}$$

(c) $\psi$ es par/impar.

$$\Rightarrow \cos\left[\int_x^a dx\, k(x) - \frac{\pi}{4}\right] = \pm \cos\left[\int_{-a}^x dx\, k(x) - \frac{\pi}{4}\right]$$

$$\Rightarrow \cos\left[-\int_x^a dx\, k(x) - \frac{\pi}{4}\right] = \cos\left[\int_x^a dx\, k(x) + \frac{\pi}{4}\right] = \pm \cos\left[\int_{-a}^x dx\, k(x) + \frac{\pi}{4}\right]$$

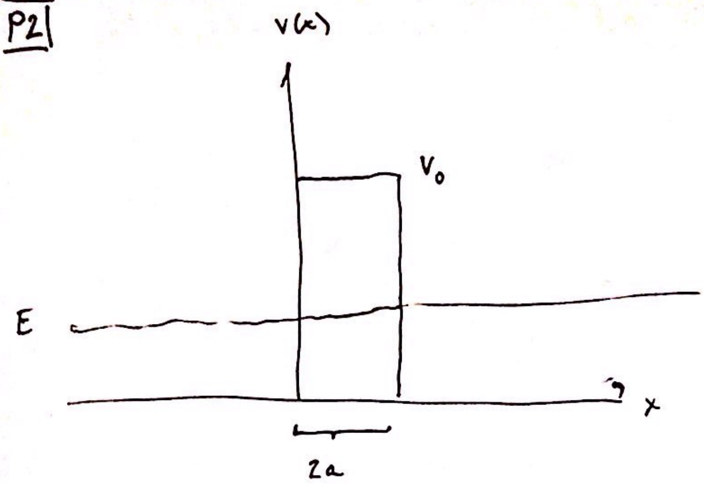
$$\Rightarrow \int_x^a dx\, k(x) + \frac{\pi}{4} = -\int_{-a}^x dx\, k(x) - \frac{\pi}{4} + (n+1)\pi, \quad n \in \mathbb{N}$$

$$\Rightarrow \boxed{\int_{-a}^a dx\, k(x) = \pi\left(n + \frac{1}{2}\right)}$$

$$\int_{-a}^a dx\, k(x) = 2 \cdot \sqrt{\frac{2m\lambda^2}{\hbar^2}} \int_0^a dx\, (a-x)^{1/2} = 2 \cdot \sqrt{\frac{2m\lambda^2}{\hbar^2}} \cdot \left. \frac{-(a-x)^{3/2}}{3/2} \right|_0^a = \frac{4}{3} a^{3/2} \sqrt{\frac{2m\lambda^2}{\hbar^2}}$$

$$= \frac{4}{3}\left(\frac{E}{\lambda^2}\right)^{3/2} \sqrt{\frac{2m\lambda^2}{\hbar^2}} \Rightarrow \boxed{E_n = \lambda^{4/3}\left(\frac{\hbar^2}{m}\right)^{1/3}\left[\frac{3\pi}{4\sqrt{2}}\left(n + \frac{1}{2}\right)\right]^{2/3}}$$

**P2**




$\psi(x) \cong \frac{A}{\sqrt{|p|}} \exp\left[\frac{-1}{\hbar}\int_c^x |p|\,dx'\right]$ dentro de la barrera.

$$T = \left|\frac{\psi(2a)}{\psi(0)}\right|^2 = \exp\left[-\frac{2}{\hbar}\int_c^{2a}|p|\,dx' + \frac{2}{\hbar}\int_c^0 |p|\,dx'\right] = \exp\left[-\frac{2}{\hbar}\int_0^{2a}|p|\,dx'\right] = \exp[-2\gamma]$$

$$\gamma = \frac{1}{\hbar}\int_0^{2a}\sqrt{2m(V_0-E)}\,dx = \frac{2a}{\hbar}\sqrt{2m(V_0-E)} \Rightarrow \boxed{T = \exp\left[\frac{-4a}{\hbar}\sqrt{2m(V_0-E)}\right]}$$ ← Analizar!

Respuesta exacta: $T = \dfrac{1}{1+\frac{V_0^2}{4E(V_0-E)}\sinh^2\gamma}$ Para WKB, $\gamma$ es grande

$$\Rightarrow \sinh^2(\gamma) \approx \frac{1}{4}e^{2\gamma} \Rightarrow T = \frac{4E(V_0-E)}{4E(V_0-E)+\frac{1}{4}V_0^2 e^{2\gamma}} \approx \frac{16E(V_0-E)}{V_0^2}e^{-2\gamma}$$

Funciona relativamente bien... comentar más!

**P4** Cond. de cuantización:

$$\pi\hbar\left(n-\tfrac{1}{2}\right) = 2\int_0^{x_2}\sqrt{2m(E-\alpha x^\nu)}\,dx$$ ⇝ Todo el problema consiste en tomar una integral!

$$\pi\hbar(n-\tfrac{1}{2}) = 2\sqrt{2mE}\int_0^{x_2}\sqrt{1-\tfrac{\alpha}{E}x^\nu}\,dx$$ Sabemos que $V(x_2) = E = \alpha x_2^\nu$. C.V. $z = \frac{\alpha}{E}x^\nu \Rightarrow x = \left(\frac{zE}{\alpha}\right)^{1/\nu}$

$$\Rightarrow dx = \left(\frac{E}{\alpha}\right)^{1/\nu}\cdot\frac{1}{\nu}z^{1/\nu-1}dz \Rightarrow \pi\hbar(n-\tfrac{1}{2}) = 2\sqrt{2mE}\left(\frac{E}{\alpha}\right)^{1/\nu}\cdot\frac{1}{\nu}\int_0^1 z^{1/\nu-1}\sqrt{1-z}\,dz = \frac{2\sqrt{2mE}}{\nu}\left(\frac{E}{\alpha}\right)^{1/\nu}\frac{\Gamma(1/\nu)\Gamma(3/2)}{\Gamma(1/\nu+3/2)}$$

$$= 2\sqrt{2mE}\left(\frac{E}{\alpha}\right)^{1/\nu}\frac{\Gamma(1/\nu+1)}{\Gamma(1/\nu+3/2)}\cdot\frac{\sqrt{\pi}}{2} = \sqrt{2\pi mE}\left(\frac{E}{\alpha}\right)^{1/\nu}\frac{\Gamma(1/\nu+1)}{\Gamma(1/\nu+3/2)}$$

$$\Rightarrow E_n^{1/\nu+1/2} = \frac{\pi\hbar(n-\frac{1}{2})\cdot\alpha^{1/\nu}\,\Gamma(1/\nu+3/2)}{\Gamma(1/\nu+1)\sqrt{2\pi m}} \Rightarrow E_n = \left[\sqrt{\frac{\pi}{2m\alpha}}\,\hbar\left(n-\tfrac{1}{2}\right)\frac{\Gamma(1/\nu+3/2)}{\Gamma(1/\nu+1)}\right]^{\frac{1}{1/\nu+1/2}}\alpha$$

$$\Rightarrow \boxed{E_n = \alpha\left[\left(n-\tfrac{1}{2}\right)\hbar\sqrt{\frac{\pi}{2m\alpha}}\,\frac{\Gamma(1/\nu+3/2)}{\Gamma(1/\nu+1)}\right]^{\frac{2\nu}{2+\nu}}}$$

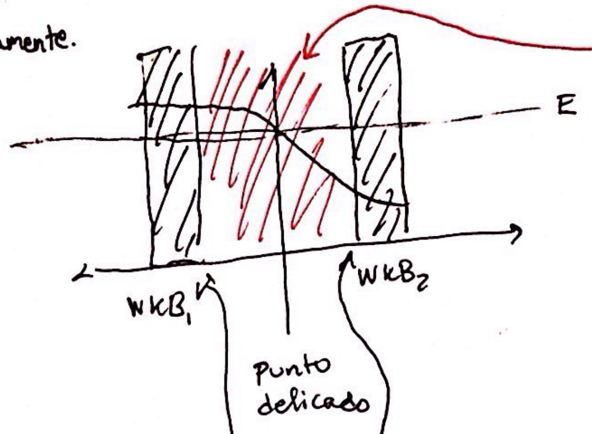
$\Gamma(1/\nu+1) = 1/\nu\,\Gamma(1/\nu)$, y adicionalmente $\Gamma(3/2) = \sqrt{\pi}/2$

$\Gamma(z) \equiv \int_0^\infty x^{z-1}e^{-x}\,dx$

(La integral original es la función beta que cumple la identidad $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ )

PROP: Sabiendo que $\Gamma(2)=1$, revisen que $E_n$ funciona bien para el oscilador!

P3 Idea: queremos ver cómo "conectar" $\psi(x)$ del lado permitido clásicamente, al permitido clásicamente.




Usar los dos lados asintóticos de las funciones de Airy para parchar!

<u>Único</u> propósito: tener $B$ y $C$ en términos de $D$.

Queremos ver cómo relacionar estos bordes. Solución: ver cómo se comporta entre medio y usar dos parches. $\rightarrow$ Airy

$$\psi_{WKB} = \begin{cases} \dfrac{D}{\sqrt{|p|}} \exp\left[\dfrac{-1}{\hbar}\displaystyle\int_x^0 |p|\, dx'\right] & x<0 \\ \dfrac{1}{\sqrt{p}}\left[B\exp\left[\dfrac{i}{\hbar}\displaystyle\int_0^x p\, dx'\right] + C\exp\left[\dfrac{-i}{\hbar}\displaystyle\int_0^x p\, dx'\right]\right] & x>0 \end{cases}$$

En la región central, $V(x) \approx E + V'(0)x \Rightarrow -\frac{\hbar^2}{2m}\psi_p'' + (\cancel{E} + V'(0)x)\psi_p = \cancel{E}\psi_p$

$$\Rightarrow \psi_p'' = \frac{2mx}{\hbar^2}V'(0)\psi_p = -\alpha^3 x\psi_p, \quad \alpha = \left(\frac{2m|V'(0)|}{\hbar^2}\right)^{1/3}$$

$\Rightarrow \psi_p(x) = a\,Ai(-\alpha x) + b\,Bi(-\alpha x)$, y el momentum acá es $p(x) = \sqrt{2m(-V'(0)x)}$

$$= \sqrt{2m|V'(0)|\,x} = \sqrt{\alpha^3\hbar^2 x} = \hbar\alpha^{3/2}\sqrt{x}$$

<u>En la región $x<0$:</u>

$\displaystyle\int_x^0 |p(x')|\,dx' = \hbar\alpha^{3/2}\int_x^0 \sqrt{-x'}\,dx' = \frac{2}{3}\hbar(-\alpha x)^{3/2}$. Además, entonces,

$\psi_{WKB} \approx D(\sqrt{\hbar}\,\alpha^{3/4}x^{1/4})^{-1} \exp\left[-\frac{2}{3}(-\alpha x)^{3/2}\right]$. Nos alejamos, entonces Airy asintóticas.

$$\psi_p \approx \frac{a}{2\sqrt{\pi}(-\alpha x)^{1/4}}\exp\left[-\frac{2}{3}(-\alpha x)^{3/2}\right] + b\cdot\frac{1}{\sqrt{\pi}(-\alpha x)^{1/4}}\exp\left[\frac{2}{3}(-\alpha x)^{3/2}\right]$$

Matching $\Rightarrow b=0$, $\dfrac{a}{2\sqrt{\pi}(-\alpha x)^{1/4}} = \dfrac{D}{\sqrt{\hbar}\,\alpha^{3/4}(-x)^{1/4}} \Rightarrow \boxed{a = 2D\sqrt{\dfrac{\pi}{\alpha\hbar}}}$

<u>Región $x>0$:</u>

$$\int_0^x p(x')\,dx' = \frac{2}{3}\hbar(\alpha x)^{3/2} \Rightarrow \psi_{WKB} = \frac{1}{\sqrt{\hbar}\,\alpha^{3/4}x^{1/4}}\left[B e^{i\frac{2}{3}(\alpha x)^{3/2}} + C e^{-i\frac{2}{3}(\alpha x)^{3/2}}\right]$$

Airy asintótico para $\alpha x >> 0$:

$$\psi_p = \frac{a}{\sqrt{\pi}(\alpha x)^{1/4}}\sin\left[\frac{2}{3}(\alpha x)^{3/2} + \frac{\pi}{4}\right] = \frac{a}{\sqrt{\pi}(\alpha x)^{1/4}}\cdot\frac{1}{2i}\left[e^{i\pi/4}e^{i\frac{2}{3}(\alpha x)^{3/2}} - e^{-i\pi/4}e^{-i\frac{2}{3}(\alpha x)^{3/2}}\right]$$

$$\Rightarrow B = \frac{a}{2i}\sqrt{\frac{\alpha\hbar}{\pi}}e^{i\pi/4}, \quad C = -\frac{a}{2i}\sqrt{\frac{\alpha\hbar}{\pi}}e^{-i\pi/4} \Rightarrow \boxed{B = -ie^{i\pi/4}D, \quad C = ie^{-i\pi/4}D}$$

Para $x>0$, $\psi_{WKB} = \dfrac{2D}{\sqrt{p}}\sin\left[\dfrac{1}{\hbar}\displaystyle\int_0^x p(x')\,dx' + \dfrac{\pi}{4}\right]$ //