$\boxed{P1}$ a. $\rho(\vec{r}) = \frac{q}{r^2}\,\delta(r-a)\,\delta(\cos(\theta))\left(\delta(\phi) + \delta(\phi - \pi/2) - \delta(\phi-\pi) - \delta(\phi - \frac{3\pi}{2})\right)$

$$q_{lm} = \int Y^*_{lm}(\theta', \phi')\, r'^{l} \rho(r')\, dv'$$

$$Y^*_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos(\theta))\, e^{-im\phi}$$

$$\Rightarrow q_{lm} = q\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, \underbrace{P_l^m(0)}_{0 \text{ si } m+l \text{ es impar}}\, a^l \underbrace{\left(1 + (-i)^m - (-1)^m - (i)^m\right)}_{0 \text{ si } m \text{ es par}}$$

$\Rightarrow q_{lm} \neq 0$ si l y m son impares

$$\Rightarrow q_{2k+1,2m+1} = q\sqrt{\frac{4k+3}{4\pi}\frac{(2k-2m)!}{(2k+2m+2)!}}\, P_{2k+1}^{2m+1}(0)\, a^{2k+1}\, 2\left(1 - i(-1)^m\right)$$

$$= 2q a^{2k+1}\, Y_{2k+1,2m+1}(\pi/2, \phi)\, e^{-i(2m+1)\phi}\left(1 - i(-1)^m\right)$$

$$q_{1,1} = -2qa(1-i)\sqrt{\frac{3}{8\pi}} = -q^*_{1,-1}$$

$$q_{3,3} = -2qa^3(1+i)\frac{1}{4}\sqrt{\frac{35}{4\pi}} = -q_{3,3}$$

$$q_{3,1} = 2qa^3(1-i)\frac{1}{4}\sqrt{\frac{21}{4\pi}} = -q^*_{3,-1}$$

b. $\rho(\vec{r}) = \frac{q}{2\pi r^2}\left(\delta(r-a)\delta(\cos(\theta)-1) + \delta(r-a)\delta(\cos(\theta)+1) - \delta(r) \right)$

$$\Rightarrow q_{lm} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} dr'\, d(\cos(\theta'))\, d\phi' \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos(\theta'))\, e^{-im\phi'} r'^{l} \rho(\vec{r}')\, r'^2$$

$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \Bigg[\frac{a^l q}{2\pi}\left(P_l^m(1) + P_l^m(-1) \right) \int_0^{2\pi} d\phi'\, e^{-im\phi'}$$

$$- \frac{q}{2\pi} \underbrace{\int_0^{2\pi} d\phi'\, e^{-im\phi'}}_{2\pi\delta_{m0}} \underbrace{\int dr'\, r'^{l}\, \delta(r')}_{\delta_{l0}} \underbrace{\int_{-1}^{1} d(\cos(\theta'))\, P_l^m(\cos(\theta'))}_{2 \text{ si } l=0} \Bigg]$$

$$\Rightarrow q_{l0} = \sqrt{\frac{2l+1}{4\pi}}\, q \left(a^l \left(1 + (-1)^l \right) - 2\delta_{l0} \right)$$

P2

$$\rho_{eff}(\vec{x}) = -\vec{p}\cdot\nabla\delta(\vec{x}-\vec{x}_0)$$

$$\Phi_{\vec{p}}(\vec{x}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^2} \quad ; \quad \vec{r} = \vec{x}-\vec{x}_0$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d\vec{x}^{3\prime}\,\rho(\vec{x}')\cdot\frac{1}{|\vec{x}-\vec{x}'|}$$

$$= \frac{-1}{4\pi\varepsilon_0}\int d\vec{x}^{3\prime}\,\vec{p}\cdot\nabla\delta(\vec{x}'-\vec{x}_0)\frac{1}{|\vec{x}-\vec{x}'|}$$

$$= \frac{-1}{4\pi\varepsilon_0}\left(\cancelto{0}{\vec{p}\,\delta(\vec{x}'-\vec{x}_0)\cdot\frac{1}{|\vec{x}-\vec{x}'|}\Big|_0^{+\infty}} - \int d\vec{x}^{3\prime}\,\vec{p}\cdot\delta(\vec{x}'-\vec{x}_0)\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\vec{p}\,\nabla\left(\frac{1}{|\vec{x}-\vec{x}_0|}\right) = \frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\frac{(\vec{x}-\vec{x}_0)}{|\vec{x}-\vec{x}_0|^3} = \frac{p\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

Podemos ver que coincida la energía:

$$W = \int d\vec{x}^{3\prime}\,\rho(\vec{x}')\,\Phi(\vec{x}')$$

$$= -\int d\vec{x}^{3\prime}\,\vec{p}\cdot\nabla\delta(\vec{x}-\vec{x}_0)\,\Phi(\vec{x}')$$

$$= -\cancelto{0}{\vec{p}\,\delta(\vec{x}-\vec{x}_0)\,\Phi(\vec{x}')\Big|_0^{\infty}} + \int d\vec{x}^{3\prime}\,\vec{p}\,\delta(\vec{x}-\vec{x}_0)\,\nabla\Phi(\vec{x}')$$

$$= \vec{p}\,\nabla(\Phi(\vec{x}_0)) = -\vec{p}\cdot\vec{E}(\vec{x}_0)$$