P1

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{|\vec{r}-\vec{r}'|}\rho(\vec{r}')dv'$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\left(1+\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos(\theta')\right)^{-1/2}$$

$$= \frac{1}{r}\left(1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos(\theta')\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos(\theta')\right)^2 + \ldots\right)$$

$$= \frac{1}{r}\left(1 + \frac{r'}{r}\cos(\theta') + \left(\frac{r'}{r}\right)^2 (3\cos^2(\theta') - 1)\cdot\frac{1}{2} + \ldots\right)$$

$$= \frac{1}{r}\sum_{n=0}\left(\frac{r'}{r}\right)^n P_n(\cos(\theta'))$$

$$\Rightarrow V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos(\theta'))\rho(r')dv'$$

$$V(\vec{r}) = \frac{1}{\varepsilon_0}\sum_{l,m}\frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\int Y^*_{lm}(\theta',\phi')r'^l\rho(\vec{r}')dv'$$

$$\rho(r,\theta) = \frac{KR}{r^2}(R-2r)\operatorname{sen}(\theta)$$

$$Q = \int \rho(r',\theta')dv' = 2\pi\int_0^\pi \operatorname{sen}^2(\theta')d\theta'\int_0^R KR\left(\frac{R}{r'^2}-\frac{2}{r'}\right)r'^2dr'.$$

$$\int_0^R (R-2r')dr' = R^2 - R^2 = 0 \Rightarrow Q = 0$$

$$\vec{p} = \int \vec{r}' \rho(\vec{r}')\, dv'$$

$$\Rightarrow \vec{p}\cdot\hat{r} = \int \hat{r}\cdot\vec{r}' \rho(\vec{r}')\, dv' = \int r'\cos(\theta')\rho(\vec{r}')\, dv'$$

$$= kR\int (r'\cos(\theta'))\left(\frac{1}{r'^2}(R-2r')\,\mathrm{sen}(\theta')\right) r'^2\,\mathrm{sen}(\theta')\, dr'\, d\theta'\, d\phi'$$

$$\int_0^{\pi} \cos(\theta')\,\mathrm{sen}^2(\theta')\, d\theta = \frac{\mathrm{sen}^3(\theta')}{3}\Big|_0^{\pi} = 0 \Rightarrow \vec{p}\cdot\hat{r} = 0$$

Cuadrupolo:

$$\int r'^2\left(\frac{3}{2}\cos^2(\theta) - \frac{1}{2}\right)\rho\, dv' = \frac{kR}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r'^2(3\cos^2(\theta')-1)\left(\frac{1}{r'^2}(R-2r')\,\mathrm{sen}(\theta')\right) r'^2\,\mathrm{sen}(\theta')\, dr'\, d\theta'\, d\phi'$$

$$r': \int_0^R r'^2(R-2r')\, dr' = \left(\frac{Rr'^3}{3} - \frac{r'^4}{2}\right)\Big|_0^R = -\frac{R^4}{6}$$

$$\theta': \int_0^{\pi}(3\cos^2(\theta')-1)\,\mathrm{sen}^2(\theta')\, d\theta' = 2\int_0^{\pi}\mathrm{sen}^2(\theta')\, d\theta' - 3\int_0^{\pi}\mathrm{sen}^4(\theta')\, d\theta'$$

$$= 2\cdot\frac{\pi}{2} - 3\cdot\frac{3\pi}{8} = \pi\left(1-\frac{9}{8}\right) = -\frac{\pi}{8}$$

$$\Rightarrow V_{cuad} = \frac{1}{4\pi\varepsilon_0 r^2}\cdot\frac{k\pi^2 R^5}{48}$$

$\boxed{P2}$ a. $\rho(\vec{r}) = \frac{\rho_0}{r^2}(d^2 - r^2)(\delta(\cos(\theta)-1) + \delta(\cos(\theta)+1))$

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}\,')G(\vec{r},\vec{r}\,')dv'$$

$$G(\vec{r},\vec{r}\,') = \sum_{l,m}\frac{4\pi}{2l+1} r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

Por simetría en $\phi \Rightarrow m=0$:

$$\Rightarrow G(\vec{r},\vec{r}\,') = \sum_l r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)P_l(\cos(\theta'))P_l(\cos(\theta))$$

$$\Rightarrow \Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_l P_l(\cos(\theta))\int \frac{\rho_0}{r'^2}(d^2 - r'^2)(\delta(\cos(\theta)-1)+\delta(\cos(\theta)+1))\cdot r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)\cdot \ldots$$

$$\ldots \cdot P_l(\cos(\theta'))r'^2 dr' d\phi' d(\cos(\theta'))$$

$$= \frac{\rho_0}{2\varepsilon_0}\sum_l P_l(\cos(\theta))(P_l(1) + P_l(-1))\underbrace{\int_0^d (d^2 - r'^2)r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)dr'}_{I_l(r)}$$

$$= \frac{\rho_0}{2\varepsilon_0}\sum_l P_l(\cos(\theta))P_l(1)(1+(-1)^l)I_l(r)$$

Para $r<d$:

$$I_\ell(r) = d^2\left(\frac{2\ell+1}{\ell(\ell+1)} + \frac{2}{\ell(\ell-2)}\left(\frac{r}{d}\right)^\ell - \frac{2}{(\ell+1)(\ell+3)}\left(\frac{r}{d}\right)^\ell\left(\frac{d}{b}\right)^{\ell+1}\right) - r^2 \cdot \frac{2\ell+1}{(\ell-2)(\ell+3)}$$

$$I_0(r) = d^2\left(\frac{1}{2} - \frac{2}{3}\left(\frac{d}{b}\right) - \ln\left(\frac{r}{d}\right)\right) + \frac{r^2}{6}$$

$$I_2(r) = \frac{5}{6}d^2 - r^2\left(\frac{7}{10} + \frac{2}{15}\left(\frac{d}{b}\right)^2 - \ln\left(\frac{r}{d}\right)\right)$$

Para $r>d$:

$$I_\ell(r) = \frac{2d^2}{(\ell+1)(\ell+3)}\left(\frac{d}{r}\right)^{\ell+1}\left(1 - \left(\frac{r}{b}\right)^{2\ell+1}\right)$$

C. Asumimos $d \ll b, r$:

$$\Phi(\vec{r}) = \frac{2\rho_0 d^3}{\varepsilon_0}\sum_{\ell\,\text{par}} \frac{P_\ell(\cos(\theta))}{(\ell+1)(\ell+3)}\left(\frac{1}{r}\left(\frac{d}{r}\right)^\ell - \frac{1}{b}\left(\frac{r}{b}\right)^\ell\left(\frac{d}{b}\right)^\ell\right)$$

$$\cong \frac{2\rho_0 d^3}{3\varepsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$$

$$Q = \int \rho(\vec{x})\,d^3x = \int \frac{\rho_0}{\cancel{r^2}}(d-r^2)\left[\delta(\cos(\theta)-1)+\delta(\cos(\theta)+1)\right]\cancel{r^2}\,dr\,d\phi\,d\cos(\theta)$$

$$= 2\pi\cdot 2\int_0^d (d^2-r^2)\,dr = \frac{8\pi}{3}\rho_0 d^3 \Rightarrow \rho_0 = 3Q/8\pi d^3$$

$$\Rightarrow \Phi \simeq \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$$