P1 La solución general para un problema con simetría en φ son los polinomios de Legendre:

$$\Phi(r,\theta) = \sum_l \left(A_l r^l + B_l r^{-l-1} \right) P_l(\cos(\theta))$$

Usamos cond. de borde para encontrar A_l y B_l:

$$\Phi(a,\theta) = \sum_l \left(A_l a^l + B_l a^{-l-1} \right) P_l(\cos(\theta)) = \begin{cases} V & \cos(\theta) \geq 0 \\ 0 & \cos(\theta) < 0 \end{cases}$$

$$\Phi(b,\theta) = \sum_l \left(A_l b^l + B_l b^{-l-1} \right) P_l(\cos(\theta)) = \begin{cases} 0 & \cos(\theta) > 0 \\ V & \cos(\theta) \leq 0 \end{cases}$$

Usamos la ortogonalidad de los polinomios de Legendre:

$$\int_{-1}^{1} dx\, P_l(x) P_m(x) = \frac{2}{2l+1} \delta_{lm}$$

$$\int_{-1}^{1} dx\, \Phi\Big|_{a,b} P_m(x) = \begin{cases} V \int_0^1 dx P_m(x) & \text{para } r=a \\ V \int_{-1}^{0} dx\, P_m(x) & \text{para } r=b \end{cases}$$

$\rightarrow \cos(\theta) = x$

$$\Longrightarrow \left(A_m a^m + B_m a^{-m-1}\right)\cdot \frac{2}{2m+1} = V\int_0^1 dx\, P_m(x)$$

$$P_m(-x) = (-1)^m P_m(x)$$

$$\left(A_m b^m + B_m b^{-m-1}\right) \frac{2}{2m+1} = V\int_{-1}^{0} dx\, P_m(x) = V(-1)^m \int_0^1 P_m(x)\,dx$$

Llamamos $N_m = \int_0^1 P_m(x)\,dx$

$$\Longrightarrow \quad A_m a^m + B_m a^{-m-1} = \frac{2m+1}{2}\cdot V N_m$$

$$A_m b^m + B_m b^{-m-1} = \frac{2m+1}{2}\cdot V(-1)^m N_m$$

$$\Longrightarrow \begin{pmatrix} a^m & a^{-m-1} \\ b^m & b^{-m-1} \end{pmatrix}\begin{pmatrix} A_m \\ B_m \end{pmatrix} = \frac{2m+1}{2}\cdot V N_m \begin{pmatrix} 1 \\ (-1)^m \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} A_m \\ B_m \end{pmatrix} = \frac{2m+1}{2}\cdot V N_m \cdot \frac{1}{b^{2m+1} - a^{2m+1}} \begin{pmatrix} (-1)^m b^{m+1} - a^{m+1} \\ (ab)^{m+1}\left(b^m + (-1)^{m+1} a^m\right) \end{pmatrix}$$

$$\Rightarrow \Phi(r,\theta) = \frac{1}{2} V \sum_{l} \frac{(2l+1) N_l}{1-\left(\frac{a}{b}\right)^{2l+1}} \left[(-1)^l \left(1 + (-1)^{l+1}\left(\frac{a}{b}\right)^{l+1}\right)\left(\frac{r}{b}\right)^l \right.$$

$$\left. + \left(1 + (-1)^{l+1}\left(\frac{a}{b}\right)^{l}\right)\left(\frac{a}{r}\right)^{l+1} \right] P_l(\cos(\theta))$$

Podemos simplificar viendo N_{2l} (los pares):

$$N_{2l} = \int_0^1 P_{2l}(x)\,dx = \frac{1}{2}\int_{-1}^{1} P_{2l}(x)\,dx = \frac{1}{2}\int_{-1}^{1} \underbrace{P_0(x)}_{P_0(x)=1} P_{2l}(x)\,dx = \frac{2\delta_{l0}}{2} = \delta_{l0}$$

$\Rightarrow$ El único término par que aporta es $l=0$, entonces $l \to 2j-1$ (impares):

$$\Phi(r,\theta) = \frac{V}{2} + \frac{V}{2}\sum_{j=1}^{\infty} \frac{(4j-1) N_{2j-1}}{1-\left(\frac{a}{b}\right)^{4j-1}} \left[-\left(1 + \left(\frac{a}{b}\right)^{2j}\right)\left(\frac{r}{b}\right)^{2j-1} \right.$$

$$\left. + \left(1 + \left(\frac{a}{b}\right)^{2j-1}\right)\left(\frac{a}{r}\right)^{2j} \right] P_{2j-1}(\cos(\theta))$$

$$N_{2j-1} = (-1)^{j+1} \frac{\Gamma\left(j-\frac{1}{2}\right)}{2\sqrt{\pi}\, j!}$$

$$\Rightarrow \Phi = \frac{V}{2} + V \sum_{j=1} \frac{(-1)^{j+1}(4j-1)\Gamma\left(j-\frac{1}{2}\right)}{4\sqrt{\pi}\, j!\left(1-\frac{a}{b}\right)^{4j-1}} \left[-\left(1+\left(\frac{a}{b}\right)^{2j}\right)\left(\frac{r}{b}\right)^{2j-1} \right.$$

$$\left. + \left(1+\left(\frac{a}{b}\right)^{2j-1}\right)\left(\frac{a}{r}\right)^{2j} \right] P_{2j-1}(\cos(\theta))$$

$b \to \infty \Rightarrow \frac{a}{b}, \frac{r}{b} \to 0$:

$$\Phi \cong \frac{V}{2} + \frac{V}{2}\left(\frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos(\theta)) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos(\theta))\right)$$

$a \to 0 \Rightarrow \frac{a}{b}, \frac{a}{r} \to 0$:

$$\Phi \cong \frac{V}{2} - \frac{V}{2}\left(\frac{3}{2}\left(\frac{r}{b}\right) P_1(\cos(\theta)) - \frac{7}{8}\left(\frac{r}{b}\right)^3 P_3(\cos(\theta))\right)$$

P2 $G(\vec{x},\vec{x}') = \sum_{l,m} \frac{4\pi}{2l+1} \cdot \frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \cdots$

$\cdots \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right) Y_l^m(\theta,\varphi)\, Y_l^{m*}(\theta',\varphi')$

$\longrightarrow \Phi = -\frac{1}{4\pi} \int_S da'\, \Phi(\vec{x}') \frac{\partial G}{\partial n'}$

$= -\frac{1}{4\pi} \int_{r'=a} \Phi(\vec{x}') \frac{\partial G}{\partial n'} a^2 d\Omega' - \frac{1}{4\pi} \int_{r'=b} \Phi(\vec{x}') \frac{\partial G}{\partial n'} b^2 d\Omega'$

Calculamos $\frac{\partial G}{\partial n'}$:

$\left.\frac{\partial G}{\partial n'}\right|_{r'=a} = -\left.\frac{\partial G}{\partial r'}\right|_{r_<=r'=a}$

$= -\frac{4\pi}{a^2} \sum_{l,m} \frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}} \left(\left(\frac{a}{r}\right)^{l+1} - \left(\frac{a}{b}\right)^{l+1} \left(\frac{r}{b}\right)^{l} \right) Y_l^m(\theta,\varphi)\, Y_l^{m*}(\theta,\varphi)$

$\left.\frac{\partial G}{\partial n'}\right|_{r'=b} = \left.\frac{\partial G}{\partial r'}\right|_{r_>=r'=b}$

$= -\frac{4\pi}{b^2} \sum_{l,m} \frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}} \left(\left(\frac{r}{b}\right)^{l} - \left(\frac{a}{b}\right)^{l} \left(\frac{a}{r}\right)^{l+1} \right) Y_l^m(\theta,\varphi)\, Y_l^{m*}(\theta,\varphi)$

Falta calcular las integrales:

$$\int_S \Phi(a,\Omega') Y_\ell^{m*}(\Omega') Y_\ell^m(\Omega) d\Omega' \quad y \quad \int \Phi(b,\Omega') Y_\ell^{m*}(\Omega') Y_\ell^m(\Omega) d\Omega'$$

tenemos que el problema es simétrico en $\varphi \Rightarrow m=0$, reemplazamos Y_ℓ^m:

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos(\theta)) e^{im\varphi}$$

$$\Rightarrow Y_\ell^0(\theta) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos(\theta)) \quad ; \quad x' = \cos(\theta') \wedge x = \cos(\theta)$$

$$\Rightarrow I_a = \frac{2\ell+1}{4\pi} \int_0^{2\pi}\int_{-1}^{1} \Phi(a,\theta') P_\ell(x') dx' d\varphi' \cdot P_\ell(x)$$

$$I_b = \frac{2\ell+1}{4\pi} \int_0^{2\pi}\int_{-1}^{1} \Phi(b,\theta') P_\ell(x') dx' d\varphi' \cdot P_\ell(x)$$

Pero $\Phi(a,\theta') = \begin{cases} V & \cos(\theta') \geq 0 \\ 0 & \cos(\theta') < 0 \end{cases} \wedge \Phi(b,\theta') = \begin{cases} 0 & \cos(\theta') > 0 \\ V & \cos(\theta') \leq 0 \end{cases}$

$$\Rightarrow \quad I_a = \frac{2\ell+1}{4\pi} \cdot 2\pi \int_0^1 V P_\ell(x') dx' \cdot P_\ell(x) = \frac{2\ell+1}{2} N_\ell \cdot P_\ell(x)$$

$$I_b = \frac{2\ell+1}{4\pi} \cdot 2\pi \int_{-1}^{0} V P_\ell(x') dx' \cdot P_\ell(x) = \frac{2\ell+1}{2} (-1)^\ell N_\ell \cdot P_\ell(x)$$

$$\Rightarrow \Phi = \sum_{l} \frac{(2l+1)VN_l}{2\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)} \left(\left(\frac{a}{r}\right)^{l+1} - \left(\frac{a}{b}\right)^{l+1} \left(\frac{r}{b}\right)^{l} \right) P_l(\cos(\theta))$$

$$+ \sum_{l} \frac{(2l+1)(-1)^{l} VN_l}{2\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)} \left(\left(\frac{r}{b}\right)^{l} - \left(\frac{a}{b}\right)^{l} \left(\frac{a}{r}\right)^{l+1} \right) P_l(\cos(\theta))$$

$$= \sum_{l} \frac{(2l+1)VN_l}{2\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)} \left(\left(1 + (-1)^{l+1} \left(\frac{a}{b}\right)^{l}\right) \left(\frac{a}{r}\right)^{l+1} \right.$$

$$\left. + (-1)^{l} \left(1 + (-1)^{l+1} \left(\frac{a}{b}\right)^{l+1}\right) \left(\frac{r}{b}\right)^{l} \right) P_l(\cos(\theta))$$