$\boxed{P1}$ $\nabla^2 \Phi = 0 \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = -4\pi f(\rho,\phi)$

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial G}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \phi^2} = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')$$

En $\phi \neq \phi'$: $\frac{\partial}{\partial \rho}\left(\rho \frac{\partial G}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial^2 G}{\partial \phi^2} = 0$

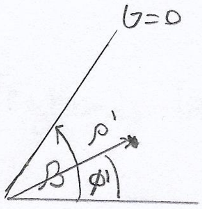
$$\rightarrow G = R(\rho)\Psi(\phi)$$

$$\Rightarrow \Psi(\phi)\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right) + \frac{R(\rho)}{\rho}\frac{\partial^2 \Psi}{\partial \phi^2} = 0$$

$$\Leftrightarrow \underbrace{\frac{\rho}{R}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right)}_{k^2} + \underbrace{\frac{1}{\Psi}\frac{\partial^2 \Psi}{\partial \phi^2}}_{-k^2} = 0$$

$$\Rightarrow \frac{\partial^2 \Psi}{\partial \phi^2} + \Psi k^2 = 0 \Rightarrow \Psi_k = A\,\mathrm{sen}(k\phi) + B\cos(k\phi)$$

Cond. de borde



$$\rightarrow G(0) = G(\beta) = 0 \Rightarrow B = 0 \wedge k_n = \frac{n\pi}{\beta}$$

$$\Rightarrow \Psi_m = A_m \operatorname{sen}\left(\frac{m\pi}{\beta}\phi\right)$$; Ahora usando la condición de simetría

$$\Rightarrow \Psi(\phi,\phi') = \Psi(\phi',\phi) \Rightarrow A_m \operatorname{sen}\left(\frac{m\pi}{\beta}\phi\right) = A_m \operatorname{sen}\left(\frac{m\pi}{\beta}\phi'\right)$$

$$\Rightarrow A_m(\phi,\phi') = C_m \operatorname{sen}\left(\frac{m\pi}{\beta}\phi'\right)$$

$$\Rightarrow \Psi_m(\phi,\phi') = C_m \operatorname{sen}\left(\frac{m\pi}{\beta}\phi\right)\operatorname{sen}\left(\frac{m\pi}{\beta}\phi'\right)$$

Superponemos todas las soluciones:

$$\Psi(\phi,\phi') = \sum_m C_m \operatorname{sen}\left(\frac{m\pi}{\beta}\phi\right)\operatorname{sen}\left(\frac{m\pi}{\beta}\phi'\right)$$

$$\Rightarrow \sum_m \left(\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) - \frac{m^2\pi^2}{\beta^2}\frac{R}{\rho}\right) C_m \operatorname{sen}\left(\frac{m\pi}{\beta}\phi\right)\operatorname{sen}\left(\frac{m\pi}{\beta}\phi'\right) = -4\pi\delta(\rho-\rho')\delta(\phi-\phi')$$

$$\Rightarrow \sum_m C_m \operatorname{sen}\left(\frac{m\pi}{\beta}\phi\right)\operatorname{sen}\left(\frac{m\pi}{\beta}\phi'\right) = \delta(\phi-\phi') \Big/ \cdot \int_0^\beta d\phi' \operatorname{sen}\left(\frac{M\pi}{\beta}\phi'\right)$$

$$\Leftrightarrow \sum_m C_m \operatorname{sen}\left(\frac{m\pi}{\beta}\phi\right)\underbrace{\int_0^\beta d\phi' \operatorname{sen}\left(\frac{m\pi}{\beta}\phi'\right)\operatorname{sen}\left(\frac{M\pi}{\beta}\phi'\right)}_{\frac{\beta}{2}\delta_{mM}} = \underbrace{\int_0^\beta d\phi' \delta(\phi-\phi')\operatorname{sen}\left(\frac{M\pi}{\beta}\phi'\right)}_{\operatorname{sen}\left(\frac{M\pi}{\beta}\phi\right)}$$

$$\Leftrightarrow C_M \cancel{\operatorname{sen}\left(\frac{M\pi}{\beta}\phi\right)}\,\beta/2 = \cancel{\operatorname{sen}\left(\frac{M\pi}{\beta}\phi\right)}$$

$$\Rightarrow C_M = 2/\beta$$

nora vemos $R(\rho)$:

Para $\rho \neq \rho$: $\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right) - k_m \frac{R}{\rho} = 0$

$\Rightarrow R(\rho) = A_m \rho^{k_m} + B_m \rho^{-k_m}$

Si $\rho < \rho'$: $R_1(\rho) = A_{m1} \rho^{k_m} + B_{m1} \rho^{-k_m}$

Si $\rho > \rho'$: $R_2(\rho) = A_{m2} \rho^{k_m} + B_{m2} \rho^{-k_m}$

$R(0) = 0 \Rightarrow B_{m1} = 0$

$R(\infty) = 0 \Rightarrow A_{m2} = 0$

$\Rightarrow$ $R_1(\rho) = A_{m1} \rho^{k_m}$

$R_2(\rho) = B_{m2} \rho^{-k_m}$

Si $\rho = \rho' \Rightarrow A_{m1} \rho'^{k_m} = B_{m2} \rho'^{-k_m}$

Ademas $G(\rho, \rho') = G(\rho', \rho) \Rightarrow A_{m1} = \gamma_m \rho'^{-k_m} \wedge B_{m2} = \gamma_m \rho'^{k_m}$

$$\Rightarrow R(\rho, \rho') = \begin{cases} \gamma_m \rho'^{-k_m} \rho^{k_m} & \text{si } \rho < \rho' \\ \gamma_m \rho'^{k_m} \rho^{-k_m} & \text{si } \rho > \rho' \end{cases}$$

$= \gamma_m \rho_<^{k_m} \rho_>^{-k_m}$; $\rho_< = \min(\rho, \rho') \wedge \rho_> = \max(\rho, \rho')$

Integramos la ec. diferencial para encontrar γ_m:

$$\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right) d\rho - k_n^2 \cancel{\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} \frac{R}{\rho} d\rho}^{\,0} = -4\pi \int_{\rho'-\varepsilon}^{\rho'+\varepsilon} \delta(\rho-\rho') d\rho \quad ; \ \varepsilon \to 0$$

$$\left(\rho \frac{\partial R}{\partial \rho}\right)\Bigg|_{\rho'+\varepsilon} - \left(\rho \frac{\partial R}{\partial \rho}\right)\Bigg|_{\rho'-\varepsilon} = -4\pi$$

$$\left(\rho \frac{\partial R}{\partial \rho}\right)\Bigg|_{\rho'+\varepsilon} = \rho' \frac{\partial R_1}{\partial \rho}\Bigg|_{\rho'} = \rho'(-k_n)\gamma_n \rho'^{k_n} \rho'^{-k_n-1} = -k_n \gamma_n$$

$$\left(\rho \frac{\partial R}{\partial \rho}\right)\Bigg|_{\rho'-\varepsilon} = \rho' \frac{\partial R_2}{\partial \rho}\Bigg|_{\rho'} = \rho'(k_n)\gamma_n \rho'^{k_n-1} \rho'^{-k_n} = k_n \gamma_n$$

$$\Rightarrow -2k_n \gamma_n = -4\pi \Rightarrow \gamma_n = \frac{2\pi}{k_n} = \frac{2\pi\beta}{\pi n} = \frac{2\beta}{n}$$

$$\Rightarrow R_n(\rho) = \frac{2\beta}{n} \rho_>^{-k_n} \rho_<^{-k_n}$$

$$\Rightarrow G(\rho,\phi;\rho',\phi') = 4\sum_{n}^{\infty} \frac{1}{n} \rho_>^{-\frac{n\pi}{\beta}} \rho_<^{\frac{n\pi}{\beta}} \operatorname{sen}\left(\frac{n\pi}{\beta}\phi\right) \operatorname{sen}\left(\frac{n\pi}{\beta}\phi'\right)$$

P2

$$\Phi = a_0 + b_0 \ln(\rho) + \sum_{n=1}^{\infty} \rho^{n} \left(A_n \operatorname{sen}(n\phi) + B_n \cos(n\phi)\right)$$
$$+ \sum_{n=1}^{\infty} \rho^{-n} \left(A_n' \operatorname{sen}(n\phi) + B_n' \cos(n\phi)\right)$$

$$\text{Si } \phi = 0 \Rightarrow \Phi = 0 \Rightarrow a_0 = b_0 = B_n = B_n' = 0$$

$$\text{Si } \phi = \beta \Rightarrow \Phi = 0 \Rightarrow n = \frac{m\pi}{\beta}$$

$$\Rightarrow \Phi = \sum_{m=1}^{\infty} \left(A_m \rho^{\frac{m\pi}{\beta}} + A_m' \rho^{-\frac{m\pi}{\beta}}\right) \operatorname{sen}(m\pi\phi/\beta)$$

$$\text{Si } \rho = 0 \Rightarrow \Phi = 0 \Rightarrow A_m a^{m\pi/\beta} + A_m' a^{-m\pi/\beta} = 0$$

$$\Rightarrow A_m' = -A_m a^{2m\pi/\beta}$$

$$\Rightarrow \Phi(\rho,\phi) = \sum_{m=1}^{\infty} A_m \left(\rho^{m\pi/\beta} - \left(\frac{a^2}{\rho}\right)^{m\pi/\beta}\right) \operatorname{sen}(m\pi\phi/\beta)$$

$$E_\rho = -\frac{\partial \Phi}{\partial \rho} = -\sum_{m=1}^{\infty} A_m \frac{m\pi}{\beta} \left(\rho^{\frac{m\pi}{\beta}-1} + \frac{a^{2m\pi/\beta}}{\rho^{m\pi/\beta+1}}\right) \operatorname{sen}(m\pi\phi/\beta)$$

$$E_\phi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} = -\sum_{m=1}^{\infty} A_m \frac{m\pi}{\rho\beta} \left(\rho^{\frac{m\pi}{\beta}} - \left(\frac{a^2}{\rho}\right)^{m\pi/\beta}\right) \cos(m\pi\phi/\beta)$$

Usando solo $m=1$:

$$\vec{E} = -\frac{A_1\pi}{\rho\beta}\left(\rho^{\pi/\beta} + \left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right) \operatorname{sen}(\pi\phi/\beta)\hat{\rho} - \frac{A_1\pi}{\rho\beta}\left(\rho^{\pi/\beta} - \left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right)\cos(\pi\phi/\beta)\hat{\phi}$$

$$\sigma(a,\phi) = \varepsilon_0 E_\rho \big|_{\rho=a} = -\frac{\varepsilon_0 \pi A_1}{\beta a}\left(2a^{\pi/\beta}\right) sen\left(\frac{\pi\phi}{\beta}\right)$$

$$\sigma(\rho,0) = \varepsilon_0 E_\phi \big|_{\phi=0} = -\frac{\varepsilon_0 \pi A_1}{\beta\rho}\left(\rho^{\pi/\beta} - \left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right)$$

$$\sigma(\rho,\beta) = \varepsilon_0 E_\phi \big|_{\phi=\beta} = \frac{\varepsilon_0 \pi A_1}{\beta\rho}\left(\rho^{\pi/\beta} - \left(\frac{a^2}{\beta}\right)^{\pi/\beta}\right)$$