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GHZ (Greenberger–Horne–Zeilinger) Theorem and GHZ States

Daniel M. Greenberger

The GHZ states (Greenberger–Horne–Zeilinger states) are a set of *entangled states* that can be used to prove the GHZ theorem, which is a significant improvement over ► *Bell's Theorem* as a way to disprove the concept of "elements of reality", a concept introduced by ► *EPR* problem (Einstein–Podolsky–Rosen) in their attempt to prove that quantum theory is incomplete. Conceding that they did not quite know what "reality" is, EPR nonetheless said that it had to contain an "element of reality" as one of its properties. This was that if one could discover a property of a system (i.e., predict it with 100% certainty) by making an experiment elsewhere, that in no way interacted with the system, then this property was an element of reality. The argument was that since one had not in any way interacted with the system, then one could not have affected this property, and so the property must have existed before one performed one's experiment. Thus the property is an intrinsic part of the system, and not an artifact of the measurement one made.

From a common-sense point of view, this proposition seems unassailable, and yet quantum theory denies it. For example, in the Bohm form of the EPR experiment, one has a particle that decays into two, that go off in opposite directions. If the original particle had ► spin 0, while each of the two daughters has spin 1/2, then if the one going to the right has its spin up, the one going to the left will have its spin down, and vice-versa. So the spin of each of the daughters is an element of reality, because if one measures the spin of the particle on the right as up, one can predict with 100% certainty that the other will be spin down, etc. EPR would conclude from this that, since we did not interfere with the particle on the left in any way, then we could not have changed its spin, and so it had to have been spin down from the moment the original particle decayed.

How can quantum theory deny this? By pointing out that since the original spin was 0, we did not have to measure the spin of the particle on the right as up or down, but we could have measured it at 90° from the vertical. Then the particle on the left would be 90° from the vertical in the opposite direction. In fact we could have measured the spin of the particle on the right in any direction, and the one on the left would be opposite it. (This is because in quantum theory, there are only two possibilities for the spin in any direction, up along that direction, or down, opposite it.) So how could the particle on the left know in which direction we were going to measure the particle on the right? Therefore the direction of its spin can not be said to exist until after the spin direction of the particle on the right is measured. Now this argument also seems unassailable, although it leads to the exact opposite conclusion from that of EPR, namely that the state of a particle cannot be defined until a measurement is made on it. And so the EPR argument has fascinated physicists since it was first given, in 1935.

Until Bell's theorem in 1964, it did not seem that the conflict here was experimentally decidable. But Bell took the EPR argument seriously, and saw that together with completeness, another postulate of EPR (all elements of reality must have some counterpart in a complete theory), it implied that there must exist some function $A(\alpha, \lambda)$, where α represents the angle along which the spin of the particle on the right is measured, and λ represents any other parameters that must be set to determine the outcome of the measurement. (These are now called ► hidden variables). The result of the experiment, the possible values for A, can only be ± 1, representing the two possible outcomes, up or down. There is a similar function representing the particle on the left, $B(\beta, \lambda)$, where β is the angle along which its spin will be measured, and the value of λ is set by nature when the particle decays.

In any given decay, one can measure the spin of the two particles along any two directions, α and β, and one will obtain the product $A(\alpha, \lambda)B(\beta, \lambda)$, as the result of the measurement. Then when one takes the result of many measurements, one will obtain an average of this product as

$$E(\alpha, \beta) = \int \mathrm{d}\lambda \rho(\lambda) A(\alpha, \lambda) B(\beta, \lambda),$$

where $\rho(\lambda)$ is some positive weighting function over the λ's, since we cannot know how often each value of λ will occur. The only limitation on this average is that when $\beta = \alpha$, then $E = -1$, since this is the condition imposed by the fact that the original particle has spin 0, and if you measure the two daughters along the same direction, the spins will be opposite each other. Equivalently, if $\beta = \alpha \pm \pi$, then $E = +1$. These two cases are known as the "perfect correlation" cases, since they represent the case where an element of reality exists, and one can predict the outcome for the product with 100% certainty. (That the function A depends only on α, and not on β, is known as the ► locality, which we have also taken to be true.)

From this form for $E(\alpha, \beta)$, as a weighted product over A and B, Bell was able to prove an inequality that the average function, E, had to satisfy, which has come to be known as the Bell inequality. Any realistic description based on the EPR elements of reality must obey this inequality. But the quantum theory expectation value violates this inequality for most sets of angles (α, β), and thus the Bell inequality established an experimental test to determine whether the EPR postulates were correct or not. The long experimental history of making the inequality experimentally useful, and the subsequent confirming of quantum theory is a fascinating tale, but it is not our concern here. Here we merely note that it is ironical that when $\beta = \alpha$, the perfect correlation case that inspired the controversy, the Bell inequality is not violated. This is because in this case it is easy to make a realistic model that explains the result. The violation occurs when one takes arbitrary angles.

The GHZ theorem concerns three particles. It considers only perfect correlations, so one does not have to take an average over many experiments. In theory one could use only a single event to prove a contradiction with the EPR result, although in practice one always needs statistics in an experiment. The GHZ theorem shows that one can construct three-particle situations in which there are perfect correlations (meaning that by measuring two particles, one can make a prediction with 100% certainty what a measurement of the third particle will yield), in which a classical, realistic interpretation will yield a particular result, while quantum mechanics predicts the exactly opposite result.

We will give a very clever version of the experiment, due to David Mermin. Consider three spin 1/2 particles. Now look at the four Hermitian operators A, B, C, D, which represent ► observables, and which are defined as

$$A = \sigma_x^1 \sigma_y^2 \sigma_y^3, \quad B = \sigma_y^1 \sigma_x^2 \sigma_y^3, \quad C = \sigma_y^1 \sigma_y^2 \sigma_x^3, \quad D = \sigma_x^1 \sigma_x^2 \sigma_x^3.$$

Here the σ's are the ► Pauli spin matrices, and the superscripts tell which particle the matrix operates on, while the subscripts define the component of the spin. All these ► operators commute with each other:

$$\begin{aligned}
&AB = \sigma_x^1 \sigma_y^2 \sigma_y^3 \sigma_y^1 \sigma_x^2 \sigma_y^3 = (\sigma_x^1 \sigma_y^1)(\sigma_y^2 \sigma_x^2)(\sigma_y^3 \sigma_y^3) = (\mathrm{i}\sigma_z^1)(-\mathrm{i}\sigma_z^2)1^3 = \sigma_z^1 \sigma_z^2,\\
&BA = \sigma_y^1 \sigma_x^2 \sigma_y^3 \sigma_x^1 \sigma_y^2 \sigma_y^3 = (\sigma_y^1 \sigma_x^1)(\sigma_x^2 \sigma_y^2)(\sigma_y^3 \sigma_y^3) = (-\mathrm{i}\sigma_z^1)(\mathrm{i}\sigma_z^2)1^3 = \sigma_z^1 \sigma_z^2,\\
&[A, B] = 0 = [A, C] = [B, C],\\
&AD = \sigma_x^1 \sigma_y^2 \sigma_y^3 \sigma_x^1 \sigma_x^2 \sigma_x^3 = 1^1(-\mathrm{i}\sigma_z^2)(-\mathrm{i}\sigma_z^3) = -\sigma_z^2 \sigma_z^3,\\
&DA = \sigma_x^1 \sigma_x^2 \sigma_x^3 \sigma_x^1 \sigma_y^2 \sigma_y^3 = 1^1(\mathrm{i}\sigma_z^2)(\mathrm{i}\sigma_z^3) = -\sigma_z^2 \sigma_z^3,\\
&[A, D] = 0 = [B, D] = [C, D].
\end{aligned}$$

(Here 1^3 means unity for particle 3, which is unity.) Thus all the operators commute and they can all be measured at the same time, and simultaneously diagonalized. Finally, their product satisfies the relation

$$\begin{aligned}ABCD &= \sigma_x^1\sigma_y^2\sigma_y^3\sigma_y^1\sigma_x^2\sigma_y^3\sigma_y^1\sigma_y^2\sigma_x^3\sigma_x^1\sigma_x^2\sigma_x^3\\ &= (\sigma_x^1\sigma_y^1\sigma_y^1\sigma_x^1)(\sigma_y^2\sigma_x^2\sigma_y^2\sigma_x^2)(\sigma_y^3\sigma_y^3\sigma_x^3\sigma_x^3) = 1^1(-\mathrm{i}\sigma_z^2)(-\mathrm{i}\sigma_z^2)1^3\\ &= -1^1 1^2 1^3 = -1.\end{aligned}$$

The above is a quantum calculation. From the point of view of a classical, realistic theory, if one measures σ_x^1, the x component of the spin for particle 1, one will get m_x^1, which $= \pm 1$. Thus, if one measures the operator ABCD, one will get

$$ABCD = (m_x^1 m_y^2 m_y^3)(m_y^1 m_x^2 m_y^3)(m_y^1 m_y^2 m_x^3)(m_x^1 m_x^2 m_x^3) = +1.$$

The product must be $= +1$, because every term appears in the product twice. But quantum mechanically, the product is -1. The reason for the difference between this and the quantum result, -1, is that even though one can make all the measurements at the same time quantum mechanically, all the spin components do not commute. (One must measure the operators A, B, C, and D, not the individual particle spins.) Thus in principle, we could make this one measurement of ABCD, and distinguish between the EPR view of reality and the quantum-mechanical one.

What are the quantum mechanical states that simultaneously diagonalize the operators A, B, C, and D? The particles cannot be in one of the states of say σ_x^1, because then one could not at the same time measure σ_y^1. So the particle cannot be in any one state, but must be in a state that is not a simple product of the states of each of the particles. In other words, it must be in an an entangled state (▶ entanglement). We call the spin states $\left|m_z^1 = +1\right\rangle \equiv \left|\uparrow^1\right\rangle$, $\left|m_z^1 = -1\right\rangle \equiv \left|\downarrow^1\right\rangle$, and simplify further by leaving out the superscripts for the particles, so that we merely denote the state $\left|\uparrow^1\right\rangle\left|\downarrow^2\right\rangle\left|\uparrow^3\right\rangle \equiv |\uparrow\downarrow\uparrow\rangle$.

Then we can use the properties of the spin states, namely

$$\begin{aligned}\sigma_x\left|\uparrow\right\rangle &= \left|\downarrow\right\rangle, \quad & \sigma_x\left|\downarrow\right\rangle &= \left|\uparrow\right\rangle,\\ \sigma_y\left|\uparrow\right\rangle &= \mathrm{i}\left|\downarrow\right\rangle, \quad & \sigma_y\left|\downarrow\right\rangle &= -\mathrm{i}\left|\uparrow\right\rangle,\\ \sigma_z\left|\uparrow\right\rangle &= \left|\uparrow\right\rangle, \quad & \sigma_z\left|\downarrow\right\rangle &= -\left|\downarrow\right\rangle,\end{aligned}$$

to verify that the state $\left|\psi_1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\uparrow\uparrow\right\rangle + \left|\downarrow\downarrow\downarrow\right\rangle\right)$ satisfies

$$\begin{aligned}A\left|\psi_1\right\rangle &= \frac{1}{\sqrt{2}}\sigma_x^1\sigma_y^2\sigma_y^3\left(\left|\uparrow\uparrow\uparrow\right\rangle + (\downarrow\downarrow\downarrow\rangle\right) = \frac{1}{\sqrt{2}}\left(-\left|\downarrow\downarrow\downarrow\right\rangle - \left|\uparrow\uparrow\uparrow\right\rangle\right) = -\left|\psi_1\right\rangle,\\ B\left|\psi_1\right\rangle &= \frac{1}{\sqrt{2}}\sigma_y^1\sigma_x^2\sigma_y^3\left(\left|\uparrow\uparrow\uparrow\right\rangle + (\downarrow\downarrow\downarrow\rangle\right) = -\left|\psi_1\right\rangle,\\ C\left|\psi_1\right\rangle &= \sigma_y^1\sigma_y^2\sigma_x^3\left|\psi_1\right\rangle = -\left|\psi_1\right\rangle,\\ D\left|\psi_1\right\rangle &= \sigma_x^1\sigma_x^2\sigma_x^3\left|\psi_1\right\rangle = +\left|\psi_1\right\rangle.\end{aligned}$$

So this state diagonalizes each of the operators A, B, C, and D. So does the state $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)$. And in fact so do all the eight states

$$
\begin{aligned}
|\psi_1\rangle &= \tfrac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)\,, & |\psi_2\rangle &= \tfrac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)\,,\\
|\psi_3\rangle &= \tfrac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle)\,, & |\psi_4\rangle &= \tfrac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle)\,,\\
|\psi_5\rangle &= \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle)\,, & |\psi_6\rangle &= \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\downarrow\rangle)\,,\\
|\psi_7\rangle &= \tfrac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle)\,, & |\psi_8\rangle &= \tfrac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\downarrow\rangle)\,.
\end{aligned}
$$

These eight entangled states are called the GHZ states, and the concept can be generalized to many particles.

The operators A, B, and C form what is called a completely commuting set of operators, and we could label the states by the eigenvalues of these operators, acting on the states, so that

$$
\begin{aligned}
&A\,|\psi_i\rangle = a_i\,|\psi_i\rangle\,, \quad B\,|\psi_i\rangle = b_i\,|\psi_i\rangle\,, \quad C\,|\psi_i\rangle = c_i\,|\psi_i\rangle\,, \quad a_i, b_i, c_i = \pm 1,\\
&|\psi_i\rangle \equiv |a_i, b_i, c_i\rangle\,.
\end{aligned}
$$

(The operator D is redundant, since $\mathrm{D} = -\mathrm{ABC}$, and $d_i = -a_i b_i c_i$.) Then

$$
\begin{aligned}
&|\psi_1\rangle = |---\rangle\,, \quad |\psi_2\rangle = |+++\rangle\,, \quad |\psi_3\rangle = |++-\rangle\,, \quad |\psi_4\rangle = |--+\rangle\,,\\
&|\psi_5\rangle = |+-+\rangle\,, \quad |\psi_6\rangle = |-+-\rangle\,, \quad |\psi_7\rangle = |-++\rangle\,, \quad |\psi_8\rangle = |+--\rangle\,.
\end{aligned}
$$

The GHZ states are entangled, non-local, and from a realistic point of view, acausal, and as we have seen, even their perfect correlations cannot be explained as elements of reality. They have been created in the laboratory, not as particles with spin 1/2, but rather as photon states, where their degrees of freedom, rather than being spin up or spin down, have been their polarization states, H or V, for horizontal or vertical, or equivalently, $+$ or $-$, for circular polarization, and in some cases their position, rather than polarization, meaning, for example, whether they were transmitted or reflected by a beam splitter. The Mermin experiment above has been performed, using photons (▶ light quantum) by the group of A. Zeilinger in Vienna (see the bibliography).

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G

Gleason's Theorem

Carsten Held

On a ► Hilbert space H, the quantum-mechanical trace formula provides a probability measure. Let {P} be the set of projection operators (► projection) on H and let, for a given statistical operator W, μ be a function from {P} into [0, 1] defined by $\mu(\mathrm{P}) = \mathrm{Tr}\,(\mathrm{P} \cdot \mathrm{W})$. Let $\{\mathrm{P}_i\} \subset \{\mathrm{P}\}$ be a countable set of mutually orthogonal projection operators. Then, $\mu(\sum_i \mathrm{P}_{|\mathrm{q}_i\rangle}) = \sum_i \mu(\mathrm{P}_{|\mathrm{q}_i\rangle})$ (countable additivity), $\mu(\mathrm{I}) = 1$, if $\sum_i \mathrm{P}_{|\mathrm{q}_i\rangle} = \mathrm{I}$, where I is the identity operator (probability of the certain event), and $\mu(\mathrm{P}_0) = 0$, where P_0 is the operator projecting on the zero space (probability of the impossible event). Hence for every particular set $\{\mathrm{P}_i\}$, μ fulfils the familiar probability axioms, i.e. is a probability measure. Obviously, μ is not a probability measure defined on the whole set {P}, since countable additivity is fulfilled, not for arbitrary, but only for mutually orthogonal elements of {P}. We have, in effect, defined a generalised probability function, a function on the lattice of projection operators such that every restriction to a Boolean sublattice is a probability measure.