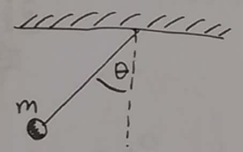
Auxiliar #2: Ecuaciones de Euler-Lagrange

$$S=\int_{t_1}^{t_2} L\,dt \quad ; \quad L=L(q_i,\dot{q}_i)$$

P.M.A.: $\delta S=0=\int_{t_1}^{t_2}\delta L\,dt=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i}\delta q_i+\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right)dt$

I.P.P. $\Rightarrow \frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=0 \qquad i\in\{1,\dots,3N\}$

Ej.:



y, x

$N=1\Rightarrow q_i\in\{x,y,z\}$

$\dot{q}_i\in\{v_x,v_y,v_z\}$

$3N-K$ coordenadas

K: constricciones

$z=cte=0$

$3-K=3-2=1$

$\hookrightarrow \theta(\dot{\theta})$

P1 Masa m 1-D, $q_i=x$, $\dot{q}_i=\dot{x}$

$$L(x,\dot{x})=\frac{1}{12}m^2(\dot{x}^2)^2+m\dot{x}^2f(x)-f^2(x)$$

Por E-L

$$\frac{\partial L}{\partial x}=m\dot{x}^2f'(x)-2f(x)f'(x) \quad ; \quad f'(x)=\frac{\partial f}{\partial x}=\frac{df}{dx}$$

$$\frac{\partial L}{\partial \dot{x}}=\frac{1}{\cancel{12}}m^2\frac{\cancel{4}\dot{x}^3}{3}+2m\dot{x}f(x)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=m^2\cdot\frac{\cancel{3}\dot{x}^2}{\cancel{3}}\cdot\ddot{x}+2m\ddot{x}f(x)+2m\dot{x}f'(x)\cdot\dot{x}$$

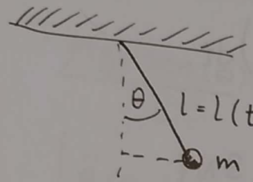
$$\Rightarrow m\dot{x}^2f'(x)-2f(x)f'(x)=m^2\dot{x}^2\ddot{x}+2m\ddot{x}f(x)+2m\dot{x}^2f'(x)$$

$$-2f(x)f'(x) = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + m\dot{x}^2 f(x)$$

$$\ddot{x}\,\left[m^2\dot{x}^2 + 2mf(x)\right] = -2f(x)\,f'(x) - m\dot{x}^2 f'(x)$$

$$\left[\;\ddot{x}(x,\dot{x}) = -\frac{m\dot{x}^2 f'(x) + 2f(x)f'(x)}{m^2\dot{x}^2 + 2mf(x)}\;\right]$$

P2



$l = l(t) = l_0\left[1 + b\,\mathrm{sen}(\omega t)\right]$

$q = \theta \;;\; \dot{q} = \dot{\theta}$

· Posición: $\vec{r} = l(t)\,\mathrm{sen}(\theta)\,\hat{\imath} - l(t)\cos(\theta)\,\hat{\jmath}$

· Velocidad: $\vec{v} = \hat{\imath}\left(\dot{l}\,\mathrm{sen}(\theta) + l\cos(\theta)\dot{\theta}\right) - \hat{\jmath}\left(\dot{l}\cos(\theta) - l\dot{\theta}\,\mathrm{sen}(\theta)\right)$

$$\vec{v}\cdot\vec{v} = v^2 = \left(\dot{l}\,\mathrm{sen}(\theta) + l\dot{\theta}\cos(\theta)\right)^2 + \left(\dot{l}\cos(\theta) - l\dot{\theta}\,\mathrm{sen}(\theta)\right)^2$$

$$v^2 = \dot{l}^2\,\mathrm{sen}^2(\theta) + \underbrace{\cancel{2l\dot{\theta}\dot{l}\,\mathrm{sen}(\theta)\cos(\theta)}}_{\to 0} + l^2\dot{\theta}^2\,\mathrm{sen}^2(\theta)$$
$$+ \dot{l}^2\cos^2(\theta) - \underbrace{\cancel{2l\dot{\theta}\dot{l}\,\mathrm{sen}(\theta)\cos(\theta)}}_{\to 0} + l^2\dot{\theta}^2\,\mathrm{sen}^2(\theta)$$

$$v^2 = \dot{l}^2\underbrace{\left(\mathrm{sen}^2(\theta) + \cos^2(\theta)\right)}_{\to 1} + l^2\dot{\theta}^2\underbrace{\left(\mathrm{sen}^2(\theta) + \cos^2(\theta)\right)}_{\to 1}$$

$$v^2 = l^2\dot{\theta}^2 + \dot{l}^2$$

↳ $K = \frac{1}{2} m v^2$ ✓

$L = K - V \;;\; V = mgy = -mgl(t)\cos(\theta)$

$L = \frac{1}{2} m\left[l^2\dot{\theta}^2 + \dot{l}^2\right] + mgl\cos(\theta)$ $\quad\begin{cases} l = l_0[1 + b\,\mathrm{sen}(\omega t)) \\ \dot{l} = l_0 b\omega\cos(\omega t) \end{cases}$

$L = (\theta, \dot{\theta}, t) \Rightarrow E = E(t) \Rightarrow$ E no se conserva $= C$

Reina Magui

$$L = \frac{1}{2} m \left[\left(l_0 [1+b\,\mathrm{sen}(\omega t)]\right)^2 \dot\theta^2 + \left(l_0 b \omega \cos(\omega t)\right)^2 \right] + m g l_0 [1+b\,\mathrm{sen}(\omega t)] \cos(\theta)$$

$$\frac{\partial L}{\partial \theta} = -m l\, \mathrm{sen}(\theta)$$

$$\frac{\partial L}{\partial \dot\theta} = m l^2 \dot\theta \; ; \; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = m l^2 \ddot\theta + 2 m \dot\theta l \cdot \dot l$$

$$\Rightarrow \cancel{\left(m l_0^2 [1+b\,\mathrm{sen}(\omega t)]^2 = m l_0^2 [\;]\right)}$$

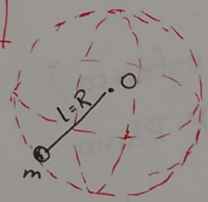
$$\frac{-\cancel{m} g \cancel{l}\, \mathrm{sen}(\theta)}{l} = \cancel{m} \cancel{l^2} \ddot\theta + 2 \cancel{m} \dot\theta \cancel{l} \frac{\dot l}{l}$$

$$\left[\ddot\theta(\theta, \dot\theta, t) = -\frac{g}{l} \mathrm{sen}(\theta) - \frac{2 \dot l \dot\theta}{l} \right]$$ Pendulo Parametrico (b)

; Si $l = cte \Rightarrow \dot l = 0$

$$\Rightarrow \ddot\theta = -\frac{g}{l} \mathrm{sen}(\theta)$$

P3



$q_i \in \{\cancel{r}, \theta, \phi\}$, $r = l$

Pos: $\vec r = l \hat r$ $\quad x = l\,\mathrm{sen}(\theta)\cos(\phi)$

$y = l\,\mathrm{sen}(\theta)\,\mathrm{sen}(\phi)$

$z = l \cos(\theta)$

Vel.: $\dot{\vec r}$: $\vec v = \dot l \hat r + l \dot\phi\, \mathrm{sen}(\theta) \hat\phi + l \dot\theta \hat\theta$

$$v^2 = l^2 \dot\phi^2 \mathrm{sen}^2(\theta) + l^2 \dot\theta^2 = l^2 (\dot\phi^2 \mathrm{sen}^2(\theta) + \dot\theta^2)$$

$L = K - V$; $L = \frac{1}{2} m l^2 (\dot\phi^2 \mathrm{sen}^2(\theta) + \dot\theta^2) + m g l \cos(\theta)$

$V = -mgl\cos(\theta)$

No hay dependencia de ϕ:

1) $\frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{d}{dt} \Big(\underbrace{\frac{\partial L}{\partial \dot\phi}}_{P_\phi}\Big) = 0$

$P_\phi = cte$ (*)

$$\Rightarrow \frac{\partial L}{\partial \dot\phi} = P_\phi = m l^2 \dot\phi\, \mathrm{sen}^2(\theta) = cte$$

(*) cons. mom angular en ϕ

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2) $\frac{\partial L}{\partial \theta} = m l^2 \dot\phi^2 \,\text{sen}(\theta)\cos(\theta) - mgl\,\text{sen}(\theta)$

$P_\theta = \frac{\partial L}{\partial \dot\theta} = m l^2 \dot\theta$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = m l^2 \ddot\theta \neq 0$

$\Rightarrow \left[\cancel{ml^2}\ddot\theta = \cancel{ml^2}\dot\phi^2 \,\text{sen}(\theta)\cos(\theta) - \frac{\cancel{mgl}}{l}\,\text{sen}(\theta) \right]$

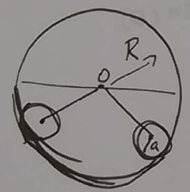
$\ddot\theta = \ddot\theta(\theta, \dot\theta, \dot\phi)$; $P_\phi = m l^2 \dot\phi \,\text{sen}^2(\theta) \Rightarrow \dot\phi(\theta) = \frac{P_\phi}{m l^2 \text{sen}^2(\theta)}$

$\Rightarrow \ddot\theta(\theta, \dot\theta) = \frac{P_\phi^2 \,\cancel{\text{sen}(\theta)}\cos(\theta)}{m^2 l^4 \text{sen}^{4\,3}(\theta)} - \frac{g}{l}\,\text{sen}(\theta)$

$\left[\ddot\theta(\theta, \dot\theta) = -\frac{g}{l}\,\text{sen}(\theta) + \frac{P_\phi^2 \cos(\theta)}{m^2 l^4 \text{sen}^3(\theta)} \right]$ (Pendulo esférico)

$P_\phi = 0 \Rightarrow$ P. Plano

P4



* C.M. Aro: $\vec{x}_{CM} = (R-a)\hat{r} \Rightarrow \vec{V}_{CM} = (R-a)\dot\theta\,\hat\theta$
* Rodar sin resbalar: $a\,d\phi = R\,d\theta$ $//\int \Rightarrow a\phi = R\theta$

· Pos: $\vec{r} = (R-a)\,\text{sen}(\theta)\,\hat\imath - (R-a)\cos(\theta)\,\hat\jmath$

Vel: $\vec{v} = (R-a)\dot\theta\cos(\theta)\,\hat\imath + (R-a)\dot\theta\,\text{sen}(\theta)\,\hat\jmath$

$\vec{v}^2 = (R-a)^2\dot\theta^2 \rightarrow K_{tras}$

$L = K - V$; $K = K_{TRAS} + K_{ROT}$ $\quad V = -mg(R-a)\cdot\cos(\theta)$

$K_{rot} = \frac{1}{2} I \omega^2$; $I = \underbrace{ma^2}_{I_{CM}} + m(R-a)^2 = m[a^2 + (R-a)^2]$; $\omega = \dot\phi$; $\dot\phi = \frac{R}{a}\dot\theta$

$$L=\frac{1}{2}m(R-a)^2\dot{\theta}^2+\frac{1}{2}m\left[a^2+(R-a)^2\right]\frac{R^2}{a^2}\dot{\theta}^2+mg(R-a)\cos(\theta)$$

$$\frac{\partial L}{\partial \theta}=-mg(R-a)\,sen(\theta);\quad \frac{\partial L}{\partial \dot{\theta}}=m(R-a)^2\dot{\theta}+m\left[a^2+(R-a)^2\right]\left(\frac{R}{a}\right)^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=\ddot{\theta}\Big[\underbrace{R-a+\left(\frac{R}{a}\right)^2\frac{a^2+(R-a)^2}{R-a}}_{\cdot c(R,a)}\Big]=-g\,sen(\theta) \iff \ddot{\theta}=\frac{-g}{c(R,a)}sen(\theta)$$

↳ Salto de paso

P.O.: $sen(\theta)\simeq\theta$

$$\ddot{\theta}=\underbrace{\frac{-g}{c}}_{-\omega_0^2}\theta \Rightarrow T_{P.O}=2\pi\sqrt{\frac{c}{g}} \sim T_p=2\pi\sqrt{\frac{L}{g}}$$