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ELEMENTS OF TURBULENCE

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INTRODUCTION

The motion of a fluid can be characterised by different regimes, depending on the specific feature that we are interested in. Thus, if we are interested in the flow dependency on the time, it could be steady (the flow does not change with time) or unsteady (the flow depends on the time). If we are interested in the changes in the space, it could be uniform (velocity and flow section do not change in space), or spatially varied. In the case of spatially varied flow, when the variation is small, the flow is gradually varied. In this case, the curvature of the streamlines is small. On the contrary, when the streamlines have strong curvature, the flow is named rapidly varied. A property of the gradually varied flows is that the pressure distribution can be considered as hydrostatic. There are many other features of the flow that we could want to study, and we could define other regimes. However, we will focus in the motion of the fluid particles. That motion can be ordered, the flow moving in "layers", and the flow regime is called laminar. On the contrary, the motion of the particles can be disorganised or random and the regime is named turbulent. Of course, there is not an abrupt change from the laminar to the turbulent regime, but it goes through a transition laminar-turbulent.

Osborne Reynolds published in 1883 an article in which he described the motion of water in a tube in which he injected dye. The experimental facility used by Reynolds and depicted in the article is shown in Fig. 1. As a curiosity, Fig. 2 shows the original apparatus used by Reynolds, currently in the University of Manchester.

For low water velocities, the dye injected in the upstream end of the tube followed a well-defined linear trajectory, as shown in Fig. 3 (taken from Reynold's paper). Reynolds called this motion "direct". For higher velocities, after some distance from the entrance of the tube, the dye "at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water", as shown in Fig. 4. But the most interesting feature was observed when the tube was illuminated with an "electric spark": the dye showed "more or less distinct curls, showing eddies", as those sketched in Fig. 5. Because of this eddy pattern, Reynolds called this motion also "sinuous". In the current language, we call

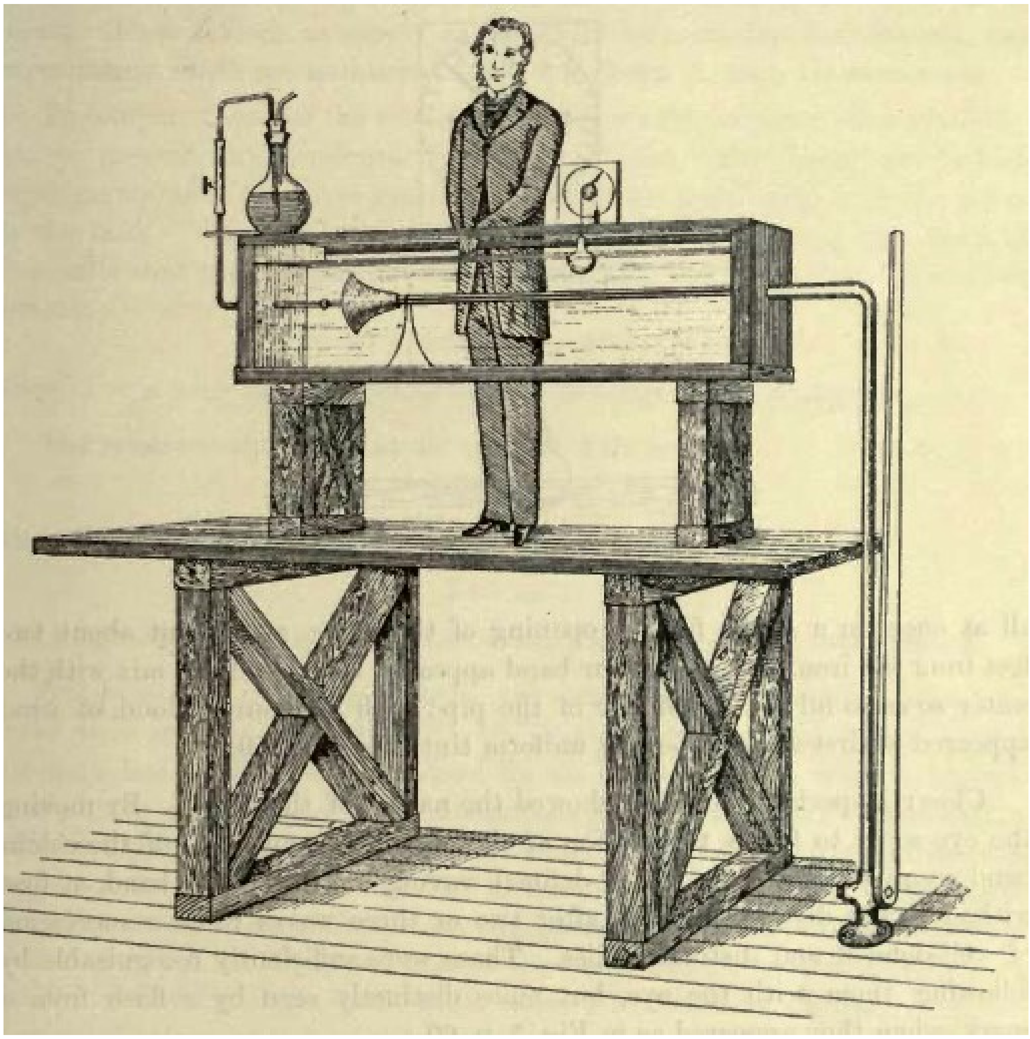


Fig. 1.- Experimental set-up of Reynold´s experiment, taken from his paper from 1883.

"laminar" and "turbulent" regimes what Reynolds called "direct" and "sinuous" motions. He also found the limits at which the regime ceased to be laminar and become turbulent was proportional to the tube diameter D, the mean flow velocity U, and inversely proportional to the kinematic viscosity ν, that is to say UD/ν. This dimensionless parameter is known as the Reynolds number:

$$Re = \frac{UD}{\nu} \tag{1}$$

The accepted values of the Reynolds numbers for the limits of the different regimes are:

- Laminar regime : $Re < 2000$

- Laminar-Turbulent transition regime : $2000 < Re < 4000$
- Turbulent regime : $Re > 4000$

Presence of eddies in the flow was not something new when Reynolds performed his

Fig. 2.- The original experimental facility used by Reynolds, now at the University of Manchester (Picture taken in May, 2015)

experiment. The eddy motions in water flows were already depicted in a series of drawings by Leonardo da Vinci, as it is shown in Fig. 6, who in the description of the motion of the surface of the water stated that it has two components, one given by the "impetus" of the main motion and another associated to the eddies. Leonardo was the first one that used the term *turbolenza* in about 1550.

Existence of eddies in the flow has an important effect in the flow resistance. Before than Reynolds' experiment was already known that for low velocities (i.e., laminar regime), the flow resistance was proportional to the mean velocity (or discharge), and for higher velocities (i.e., turbulent regime) it was proportional to the square of the mean velocity.

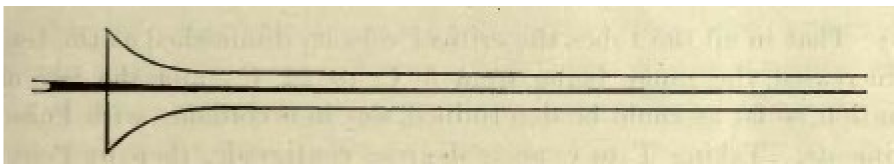


Fig. 3.- Flow pattern of the dye at low velocities. "Direct" motion or laminar regime.

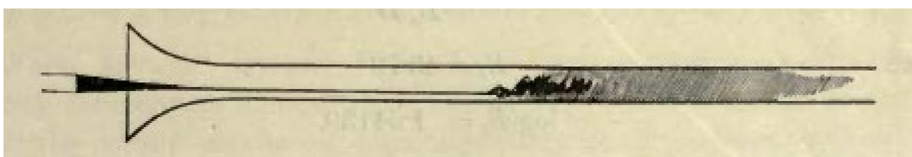


Fig. 4.- Flow pattern of the dye at high velocities. "Sinuous" motion or turbulent regime.

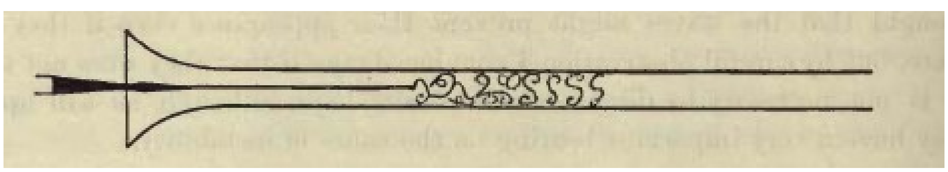


Fig. 5.- Flow pattern of the dye in the "sinuous" motion or turbulent regime when illuminated with a sparkling light. Reynolds observed the eddy or curly motion of the fluid.

WHAT IS TURBULENCE?

It is difficult to answer this question. As the velocity is continuously changing (o fluctuating) in time, we can say that it is an unsteady phenomenon. However, usually we are not interested in the instantaneous values of the velocity (or pressure) of a turbulent flow, but we want to know some time averaged values. Thus, the continuous variation in

Fig. 6.- Drawing of Leonardo da Vinci showing the presence of eddies in a flow with free Surface.

time of the flow properties has originated a statistical description of the flow in terms of temporal mean values.

Due to the complexity of the turbulent flows, most of what we know about them comes from experiments. By means of experiments, using visualization techniques or direct measurements of the variables that we are interested in (usually, velocity and pressure), we can get a general description of the processes involved in the turbulent flows. The use of the experimental approach does not mean that theories have not been developed, or that are of lesser value than the experiments. On the contrary, the theoretical analysis guide us regarding the variables that need to be measured for a better understanding of the turbulence phenomena. Obviously, as in all branches of science, the interaction between theory and experiments is necessary, and both of them help to increase our understanding of the turbulence. It is important to remark that turbulence is still an open problem. That is to say that currently the problem has not been solved starting from the principles of the physics. Thus, any theory on turbulence will always rely on some experimental data.

Up to now, there is not a definition of turbulence that can describe it completely. Von Karman (1937) defined it as "an irregular motion which in general makes its appearance in fluids, gaseous or liquid, when they flow past solid surfaces or even when neighbouring streams of the same fluid flow past or over one another". According to Hinze (1959) "turbulent fluid motion is an irregular condition of the flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned". This definition has a couple of consequences: the variation must be space-temporal, and it can be studied by means of statistical analysis.

We can look for the definition given by different dictionaries, for example:

- English
  - Oxford Dictionary (https://en.oxforddictionaries.com/): "Violent or unsteady movement of air or water, or of some other fluid"
  - Dictionary.com (http://www.dictionary.com): "The haphazard secondary motion caused by eddies within a moving fluid"
  - Merriam-Webster Dictionary: "Departure in a fluid from a smooth flow"

- Italian
  - Dizionario Italiano (www.dizionario-italiano.it): "Fenomeno per cui in un fluido liquido o gassoso in moto, in determinate condizioni, il moto delle particelle elementari cessa di essere regolare e diventa soggetto a forti fluttuazioni della velocità e a moti vorticosi e a mancanza di regolarità nella traiettoria delle particelle"
  - Grande Dizionario Italiano di Aldo Gabrielli (www.grandidizionari.it): "Moto irregolare generalmente rilevabile nei fluidi"
  - Dizionari di Italiano Corriere della Sera (http://dizionari.corriere.it/dizionario_italiano): "Moto disordinato di un fluido, con formazione di vortici"

- Spanish
  - Diccionario de la Real Academia Española (www.rae.es): "Zona en que se desarrolla un movimiento turbulento".
  - Turbulento: "Dicho del movimiento: Propio de un fluido en el que la presión y la velocidad fluctúan muy irregularmente en cada punto, con la consiguiente formación de remolinos."

As it can be seen, even the definition of the word turbulence in different languages and dictionaries is not unique. However, from the definitions presented before we can group words that transmit more or less the same meaning. For example, we find:

- "Violent or unsteady movement", "forti fluttuazioni della velocità",
- "Il moto delle particelle elementari cessa di essere regolare", "Moto irregolare", "Moto disordinato", "la presión y la velocidad fluctúan muy irregularmente"
- "Eddies", "moti vorticosi", "Moto …con formazione di vortici", "movimiento …con … formación de remolinos"

We note that the concept of eddies, vortici, remolinos, appears in several definitions. As we will see later, this is a very important concept in turbulence, used to model turbulent flows.

Given the difficulties to define turbulence, it is easier to give some characteristics of turbulent motion. Thus, according to Tennekes and Lumley (1972), the following characteristics can be mentioned:

Turbulence characterises a kind of flow: Turbulence is not a property of a fluid, it is a kind of flow. For this reason, the most important characteristics of turbulent flows are not controlled by the molecular properties of the flow.

Large Reynolds numbers: Turbulent flows occur at high Reynolds numbers. Turbulence often is originated by the instability of a laminar flow when the Reynolds number surpasses a critical value. In turbulent flows, the effect of the inertia of the fluid motion overcomes the stabilizing effect of the viscous forces.

Continuum: Turbulence is a continuum phenomenon, governed by the equations of fluid mechanics. The smallest length scales that can arise in a turbulent flow are much larger than any molecular length scale.

Irregularity: As it is observed in Reynolds' experiment, the trajectory of a fluid particle is highly irregular, making practically impossible to describe it by means of deterministic analysis, and statistical methods are used.

Diffusivity: Turbulent flows are highly diffusive. For this reason, exchange rates of mass, momentum and energy are much higher than those associated to laminar flows. From the applications point of view, this is the most important characteristic of turbulent flows.

Dissipation: Turbulent flows always dissipate energy. The work performed by the viscous stresses when deform a fluid element increases the internal energy of the fluid in detriment of the turbulent kinetic energy. Turbulence requires a continuous supply of energy in order to sustain the viscous losses. If energy is not supplied, the turbulence decays.

Tri-dimensional fluctuations of vorticity: Turbulence is rotational and tri-dimensional. It is characterized by large vorticity fluctuations. An important mechanism for the maintenance of the vorticity is "vortex stretching". Turbulent fluctuations of vorticity cannot be sustained by themselves in two-dimensional flows.

REVIEW OF THE BASIC EQUATIONS OF FLUID MECHANICS

General speaking, there two approaches to analyse fluid flows: the integral approach and the differential approach. In the integral approach we use a control volume to characterise the fluid flow (or the effect of the fluid flow) in a finite region of the space. Some gross quantities are obtained when this approach is used: mean velocity trough a section, force acting on a surface, total energy exchange, etc. On the contrary, in the differential approach the information that we get is in a point within the flow domain.

Up to now, the main principles of the physics applied to the analysis of the fluid flows using the integral approach. Those principles are: conservation of matter, Newton´s second law, and the first principle of thermodynamics. Without entering in details, the resulting equations that describes those principles, when applied to an incompressible fluid, are:

Continuity equation: It describes mathematically the law of conservation of the matter. As the amount of matter remains constant (this principle is valid in classical mechanics), for a fluid with constant density, the volume rate of fluid that enters to the control volume minus the volume rate that exits is equal to the variation of volume of fluid in the control volume, i.e.:

$$\frac{dV}{dt} = Q_i - Q_o \tag{1}$$

In Eq. 1, V represents the volume of the fluid inside the control volume, t is the time, Q is the volume rate or discharge, the sub-index i indicates the input to the control volume and the sub-index o the output.

If the discharge Q goes through a section with an area normal to the flow equal to A, the average velocity is defined as

$$v = \frac{Q}{A} \tag{2}$$

In a steady state flow there is not variation in time, $dV/dt = 0$, and Eq. 1 is simplified to:

$$Q_i = Q_o \tag{3}$$

Eq. 3 indicates that the flow that enters to the control volume is equal to the output flow.

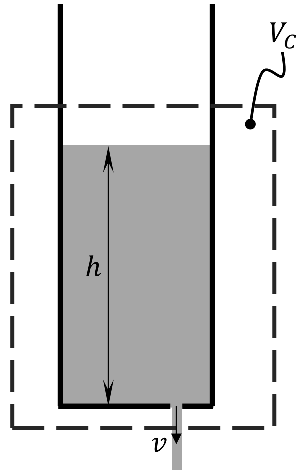


Fig. 7.- Flow through an orifice

The classical example where Eq. 1 is applied corresponds to the flow through and orifice in a tank. Referred to Fig. 7, the question is to determine the variation of the water depth (h) in the tank in function of the time, if initially the flow depth is h_0. The transverse area of the tank is A_T and the area of the orifice is A_o. This simple example is repeated here to stress two isues:

i) The volume of fluid not necessarily is equal to the control volume, V_C, depicted as the segmented line in Fig. 7.
ii) Eq. 1 by itself is not enough to solve the problem.

The volume of fluid (V) in the control volume V_C is $V = hA_T$. There is not flow rate entering to V_C, so $Q_i = 0$, and the flow rate exiting the control volume is $Q_o = vA_o$. Thus, Eq. 1 can be written as:

$$\frac{d}{dt}(hA_T) = -vA_o \tag{4}$$

We cannot go further with Eq. 4 without additional information. Up to now, we are having two unknowns: h and v, but only one equation (Eq. 4). We need another equation involving h and v. That equation will be provided by the energy equation. But it is possible to get a result if we remember the experimental work by Torricelli, who obtained:

$$v = \sqrt{2gh} \tag{5}$$

Replacing Eq. 5 in Eq. 4, we obtain:

$$\frac{dh}{dt} = -\frac{A_o}{A_T}\sqrt{2gh} \tag{6}$$

Eq. 6 can be easily integrated. The integration constant is obtained from the initial condition $t = 0$, $h = h_0$.

Energy general equation: It is obtained from the first principle of the thermodynamics, which states that the variation of energy in a system of particles, ΔE, is equal to the heat supplied to the system, $\Delta \hat{Q}$, minus the mechanical work done by the system, ΔW. The rate of variation of energy can be written as:

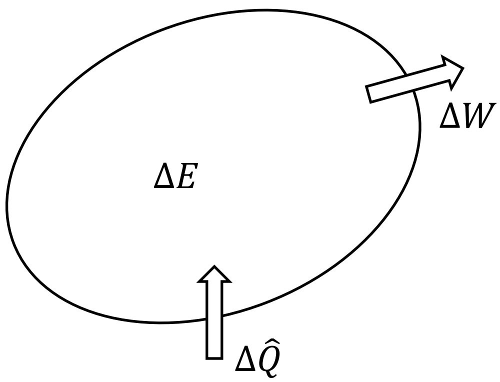


Fig. 8.- First principle of thermodynamics: Variation of energy

$$\frac{dE}{dt} = \frac{d\hat{Q}}{dt} - \frac{dW}{dt} \tag{7}$$

The energy of the fluids particles is given by three components:

$$E = U + E_P + E_C \tag{8}$$

where U is the internal energy, E_P is the potential energy and E_C is the kinetic energy. They are expressed as:

$$U = mu \qquad E_P = mgz \qquad E_C = \frac{1}{2}mv^2 \tag{9}$$

where m is the mass, u is the specific internal energy, g is the acceleration due to gravity, z is the vertical distance from an arbitrary reference level, and v the velocity. The specific energy (defined as energy per unit mass, $e = E/m$), of the system is:

$$e = u + gz + \frac{v^2}{2} \tag{10}$$

It can be shown that the variation of energy of the system can be expressed as the change of energy per unit time of the fluid in the control volume, plus the flow of energy through the surfaces S_C that define V_C. Mathematically, it is written as:

$$\frac{\partial}{\partial t}\int_{V_C} \rho e dV + \int_{S_C} \rho e \vec{v} \cdot \hat{n} dS = \frac{d\hat{Q}}{dt} - \frac{dW}{dt} \tag{11}$$

The work W can be divided in two main components: 1) the work that the fluid has to do when it flows (it has to overcome the forces arising from the pressure and shear stresses), and 2) the external work (also called "shaft work"). The external work is that done to move the shaft of a turbine (work towards the exterior of the system, work done *by* the fluid, positive), or that supplied by a pump (work towards the interior of the system, work done *on* the fluid, negative). Thus:

$$W = W_E + W_p + W_\tau \tag{12}$$

where the sub-indices E, p and τ stand for "external", "pressure" and "shear stresses", respectively.

It is possible to work with the term associated to the pressure:

$$W_p = \int dW_p = \int \vec{F}_p \cdot d\vec{r} = \int p dS \hat{n} \cdot d\vec{r} \tag{13}$$

$$\frac{dW_p}{dt} = \int p dS \hat{n} \cdot \frac{d\vec{r}}{dt} = \int_{S_C} p \vec{v} \cdot \hat{n} dS \tag{14}$$

The work associated to the shear stresses demands to know an expression for the shear stress, and we will work on it later. Thus, Eq. 11 becomes:

$$\frac{\partial}{\partial t}\int_{V_C} \rho e dV + \int_{S_C} \rho e \vec{v} \cdot \hat{n} dS = \frac{d\hat{Q}}{dt} - \frac{dW_E}{dt} - \int_{S_C} p \vec{v} \cdot \hat{n} dS - \frac{dW_\tau}{dt} \tag{15}$$

Introducing the expression for the specific energy e in the second term of the left hand side of Eq. 15:

$$\frac{\partial}{\partial t}\int_{V_C} \rho e dV + \int_{S_C} \rho\left(u + gz + \frac{v^2}{2}\right)\vec{v}\cdot\hat{n}dS = \frac{d\hat{Q}}{dt} - \frac{dW_E}{dt} - \int_{S_C} p\vec{v}\cdot\hat{n}dS - \frac{dW_\tau}{dt} \quad (16)$$

Joining the terms that are integrated over S_C:

$$\frac{\partial}{\partial t}\int_{V_C} \rho e dV + \int_{S_C} \rho\left(u + gz + \frac{p}{\rho} + \frac{v^2}{2}\right)\vec{v}\cdot\hat{n}dS = \frac{d\hat{Q}}{dt} - \frac{dW_E}{dt} - \frac{dW_\tau}{dt} \quad (17)$$

We can see that the sum of Bernoulli was formed in the second term of Eq. 17, and it can be written as:

$$\frac{\partial}{\partial t}\int_{V_C} \rho e dV + \int_{S_C} \rho g\left(\frac{u}{g} + B\right)\vec{v}\cdot\hat{n}dS = \frac{d\hat{Q}}{dt} - \frac{dW_E}{dt} - \frac{dW_\tau}{dt} \quad (18)$$

where

$$B = z + \frac{p}{\rho g} + \frac{v^2}{2g} \quad (19)$$

Eq. 17 (or Eq. 18) is called the "energy general equation". It can be applied to any flow regime. Its only limitation is given by the potential energy should derive from a gravitational field (Eq. 9).

Let's simplify Eq. 18 assuming steady flow and considering a stream tube as control volume, as shown in Fig. 9. The control volume has three surfaces: section 1 (entrance), section 2 (exit) and the mantle, with surface areas S_1, S_2 and S_m, respectively. Thus, the surface S_C that defines the control volume V_C can be written as $S_C = S_1 + S_2 + S_m$. As the mantle is tangent to the velocity vectors, there is not flow through its surface, and the integral over the surface of Eq. 18 is reduced to:

$$\int_{S_C} \rho g\left(\frac{u}{g} + B\right)\vec{v}\cdot\hat{n}dS = \int_{S_1} \rho g\left(\frac{u}{g} + B\right)\vec{v}\cdot\hat{n}dS + \int_{S_2} \rho g\left(\frac{u}{g} + B\right)\vec{v}\cdot\hat{n}dS \quad (20)$$

For simplicity, we can take that the velocity vector at sections 1 and 2 is normal to the surface, that is to say that in section 1: $\vec{v} = v_1(-\hat{n}_1)$, and in section 2: $\vec{v} = v_2\hat{n}_2$. Thus, Eq. 20 becomes:

$$\int_{S_C} \rho g\left(\frac{u}{g} + B\right)\vec{v}\cdot\hat{n}dS = -\int_{S_1} \rho g\left(\frac{u}{g} + B\right)v_1 dS + \int_{S_2} \rho g\left(\frac{u}{g} + B\right)v_2 dS \quad (21)$$

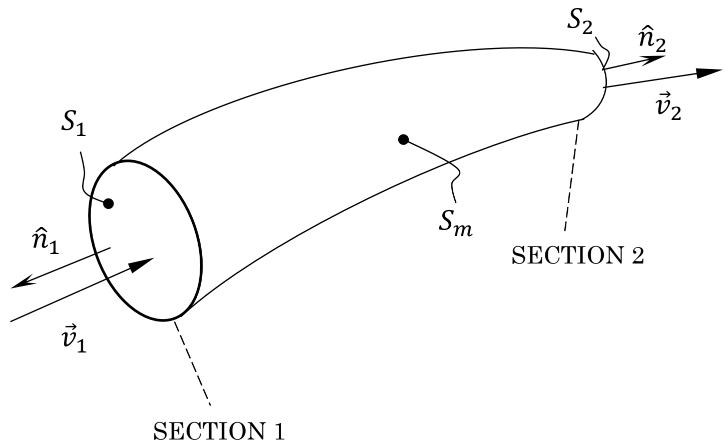


Fig. 9.- Application of the energy general equation to a streamtube

Now, we will do a strong assumption (but we will be able to overcome it, so it will not be a problem in the future): let's assume that the flow properties are constant at each section of the stream flow (they are constant in the section, but can change from one section to another). If this is the case, u, p/ρ and v, do not depend on dS and we can take the $\rho g(u+B)v$ out of the integral:

$$\int_{S_C} \rho g\left(\frac{u}{g}+B\right)\vec{v}\cdot\hat{n}dS = -\rho g\left(\frac{u_1}{g}+B_1\right)v_1\int_{S_1} dS + \rho g\left(\frac{u_2}{g}+B_2\right)v_2\int_{S_2} dS \qquad (22)$$

$$\int_{S_C} \rho g\left(\frac{u}{g}+B\right)\vec{v}\cdot\hat{n}dS = -\rho g\left(\frac{u_1}{g}+B_1\right)v_1 S_1 + \rho g\left(\frac{u_2}{g}+B_2\right)v_2 S_2 \qquad (23)$$

But $v_1 S_1$ is the discharge at section 1 which, by continuity, is equal to the discharge at section 2, i.e.: $v_1 S_1 = v_2 S_2 = Q$, and the integral over the surfaces is reduced to a simpler expression:

$$\int_{S_C} \rho g\left(\frac{u}{g}+B\right)\vec{v}\cdot\hat{n}dS = \rho g Q\left(\frac{u_2-u_1}{g}+B_2-B_1\right) \qquad (24)$$

Thus, for a steady flow through a stream tube with one entrance and one exit (as in Fig. 9), the general equation of the energy is written as:

$$\rho g Q\left(\frac{u_2-u_1}{g}+B_2-B_1\right) = \frac{d\hat{Q}}{dt} - \frac{dW_E}{dt} - \frac{dW_\tau}{dt} \qquad (25)$$

For flows of liquids (the most common in civil and environmental engineering), the variation of internal energy is negligible, and we can write:

$$\rho g Q(B_2 - B_1) = \frac{d\hat{Q}}{dt} - \frac{dW_E}{dt} - \frac{dW_\tau}{dt} \tag{26}$$

For adiabatic flows (no heat exchange) of ideal fluids without work done by the fluid, we obtain that the Bernoulli remains constant. :

$$B_2 = B_1 = Const. \tag{27}$$

At this point, an objection could be made to the development of Eq. 26 when it is applied to real fluids. If the stream tube is considered to be a pipe, due to the fluid viscosity, the velocity of the fluid particles in contact with the wall of the pipe is zero and maximum at the axis of the pipe, violating the assumption of constant velocity made to take the Bernoulli out of the surface integral and obtain Eq. 25, which contains a term associated to the shear stress. In order to avoid this inconsistency, the Bernoulli equation (Eq. 19) needs to be modified. Let's consider a flow with parallel streamlines (as the flow in a pipe of constant diameter or the uniform flow in a channel) and analyse the term that contains B in Eq. 18. As it was said, the variation of internal energy in liquids can be neglected and we will leave it out. In a given section, we have:

$$\int_S \rho g B \vec{v} \cdot \hat{n} dS = \int_S \rho g \left(z + \frac{p}{\rho g} + \frac{v^2}{2g} \right) v dS \tag{28}$$

In any flow with parallel streamlines, the sum $z + p/(\rho g)$ remains constant. In particular, we can evaluate that sum at the gravity centre (G) of the section:

$$\int_S \rho g \left(z + \frac{p}{\rho g} + \frac{v^2}{2g} \right) v dS = \int_S \rho g \left(z + \frac{p}{\rho g} \right) v dS + \int_S \rho g \frac{v^2}{2g} v dS \tag{29}$$

$$\int_S \rho g \left(z + \frac{p}{\rho g} + \frac{v^2}{2g} \right) v dS = \rho g \left(z + \frac{p}{\rho g} \right) \int_S v dS + \rho g \int_S \frac{v^2}{2g} v dS \tag{30}$$

$$\int_S \rho g B v dS = \rho g \left(z_G + \frac{p_G}{\rho g} \right) \int_S v dS + \rho g \int_S \frac{v^2}{2g} v dS \tag{31}$$

As the velocity varies from zero to a maximum value, we want to express the Bernoulli in terms of the mean velocity $\bar{v} = Q/A$, where A is the area of the surface S. Because B depends on the square of v, the effect of using $\bar{v}$ in the Bernoulli is taking into account by means of a coefficient α affecting the term with$\bar{v}^2$. Calling B^* the Bernoulli based upon $\bar{v}$:

$$B^* = z + \frac{p}{\rho g} + \alpha \frac{\bar{v}^2}{2g} \tag{32}$$

We want to have:

$$\int_S \rho g B v dS = \int_S \rho g B^* \bar{v} dS \tag{33}$$

Replacing Eq. 31 in the left side of Eq. 33 and Eq. 32 in the integral of the right side:

$$\rho g \left(z_G + \frac{p_G}{\rho g} \right) \int_S v dS + \rho g \int_S \frac{v^2}{2g} v dS = \int_S \rho g \left(z + \frac{p}{\rho g} + \alpha \frac{\bar{v}^2}{2g} \right) \bar{v} dS \tag{34}$$

$$\begin{aligned} \rho g \left(z_G + \frac{p_G}{\rho g} \right) \int_S v dS + \rho g \int_S \frac{v^2}{2g} v dS \\ = \rho g \left(z_G + \frac{p_G}{\rho g} \right) \int_S \bar{v} dS + \rho g \int_S \alpha \frac{\bar{v}^2}{2g} \bar{v} dS \end{aligned} \tag{35}$$

But

$$\int_S v dS = \int_S \bar{v} dS = Q \tag{36}$$

Using Eq. 36, Eq. 35 is reduced to:

$$\int_S \frac{v^2}{2g} v dS = \int_S \alpha \frac{\bar{v}^2}{2g} \bar{v} dS \tag{37}$$

From where we can obtain an expression for the coefficient α that takes into account that the velocity is not constant in any section of the stream tube:

$$\alpha = \frac{\int_S v^3 dS}{\bar{v}^3 A} \tag{38}$$

The coefficient α is called Coriolis coefficient. It is easy to see that in order to compute α from Eq. 38, the velocity distribution has to be known, which is not always possible, and many times it is determined from experiments or field measurements. For turbulent flow in rectilinear pipes $\sim 1.03 - 1.1$, depending on Re. As α is close to 1, the value $\alpha = 1$ is commonly used. For a laminar flow in a pipe, $\alpha = 2$.

Rigorously, Eq. 26 should be written as:

$$\rho g Q(B_2^* - B_1^*) = \frac{d\hat{Q}}{dt} - \frac{dW_E}{dt} - \frac{dW_\tau}{dt} \tag{39}$$

with

$$B^* = z + \frac{p}{\rho g} + \alpha \frac{\bar{v}^2}{2g} \tag{32}$$

However, it is customary that the bar $^{-}$ over the velocity and the asterisk * of the Bernoulli are not written, and Eq. 32 becomes:

$$B = z + \frac{p}{\rho g} + \alpha \frac{v^2}{2g} \tag{40}$$

As most of the applications deal with turbulent flow, $\alpha = 1$ is used, Eq. 32 ends written as:

$$B = z + \frac{p}{\rho g} + \frac{v^2}{2g} \tag{41}$$

A word of caution is necessary here: Eq. 41 is easily confused with Eq. 19, although they are different. Eq. 41 can be applied to a real fluid with non-uniform velocity profile, whereas Eq. 19 is restricted to uniform velocity profile, something that cannot be attained by flows of real fluids with solid boundaries. Following the common usage, we will write B, although we will be working with B^*. Similarly, most of the time we will not write the Coriolis coefficient because the flow will be turbulent.

If there is a hydraulic machine like a pump or a turbine, dW_E/dt corresponds to the power of the machine, P. We use to write the power divided by $(\rho g Q)$. For an ideal flow without heat exchange:

$$(B_2 - B_1) = -\frac{1}{\rho g Q}\frac{dW_E}{dt} = -\frac{P}{\rho g Q} \tag{42}$$

As we already mentioned, for a pump $P < 0$ and for a turbine $P > 0$. Usually the notation $\Delta B = |P|/(\rho g Q)$ is used, with the absolute value of the power. The sign (+) for a turbine or (–) for a pump should be written explicitly in the equation.

Let's consider now the adiabatic flow of a real fluid, without external work. The term $-dW_\tau/dt$ is the power dissipated by viscous effects. It is customary in hydraulics to work with the power divided by $(\rho g Q)$. In this case, Eq. 39 becomes:

$$(B_2 - B_1) = -\frac{1}{\rho g Q}\frac{dW_\tau}{dt} \tag{43}$$

The dissipated energy per unit weight of the fluid is denoted by Λ, which is defined as:

$$\Lambda \equiv \frac{1}{\rho g Q}\frac{dW_\tau}{dt} \tag{44}$$

Thus, Eq. 43 is written as:

$$B_2 = B_1 - \Lambda \tag{45}$$

The energy loss is usually divided in two terms: one associated to friction and other to singularities in the flow line. Let's restrict our attention to the friction loss. The energy loss per unit length is denoted by J, defined as:

$$J = -\frac{dB}{dx} \tag{46}$$

Thus, the friction loss in a pipe of length L is given by $\Lambda = JL$.Computation of J is generally performed using the Darcy-Weisbach equation, which requires to know the friction factor f. For a cylindrical pipe of diameter D:

$$J = f\frac{1}{D}\frac{v^2}{2g} \tag{47}$$

In general, f depends of the flow regime, given by the Reynolds number, Re, and the relative roughness of the pipe, ε/D, where ε is the size of the roughness. It can be computed analytically for laminar flows, resulting $f = 64/Re$. For turbulent flows, f can be determined semi-analytically, as we will see later. When Eq. 47 is applied to non-circular conduits, the diameter D should be replaced by the four times the hydraulic radius, defined as $R_H = A/\chi$, where A is the flow area and χ is the wetted perimeter (i.e., $D = 4R_H$)

Momentum theorem: It corresponds to the application of Newton's second law to a fluid system:

$$\frac{d}{dt}(m\vec{v}) = \vec{F} \tag{48}$$

For simplicity, let's apply Eq. 48 to the same control volume used in the derivation of the energy general equation (Fig. 9). It can be shown that the term that indicates the variation of momentum in Eq. 48, when apply to a fluid, becomes:

$$\frac{d}{dt}(m\vec{v}) = \frac{\partial}{\partial t}\int_{V_C} \rho\vec{v}\, dV + \int_{S_C} \rho\vec{v}\,\vec{v}\cdot\hat{n}dS \tag{49}$$

The integral over the surfaces of the control volume is split in three terms:

$$\int_{S_C} \rho\vec{v}\,\vec{v}\cdot\hat{n}dS = \int_{S_1} \rho\vec{v}\,\vec{v}\cdot\hat{n}dS + \int_{S_m} \rho\vec{v}\,\vec{v}\cdot\hat{n}dS + \int_{S_2} \rho\vec{v}\,\vec{v}\cdot\hat{n}dS \tag{50}$$

Considering that there is no flow through the mantle and that $\vec{v}_1\cdot\hat{n}_1 < 0$, $\vec{v}_2\cdot\hat{n}_2 > 0$, constant properties at each transverse section of the stream tube, and $v_1S_1 = v_2S_2 = Q$, Eq. 48 can be written as:

$$\frac{\partial}{\partial t}\int_{V_C} \rho\vec{v}\, dV + \rho Q(\vec{v}_2 - \vec{v}_1) = \vec{F} \tag{51}$$

For a steady flow, Eq. 51 is simplified to:

$$\rho Q(\vec{v}_2 - \vec{v}_1) = \vec{F} \tag{52}$$

As it happened with the energy general equation, Eq. 52 is valid only for uniform velocity profiles. In order to take into account the non-uniformity of the velocity profile imposed by boundaries in flow of real fluids, the cross section mean velocity is used in conjunction with a coefficient β and Eq. 53 becomes:

$$\rho Q(\beta_2\vec{v}_2 - \beta_1\vec{v}_1) = \vec{F} \tag{52}$$

The coefficient β is named Boussinesq coefficient and it is given by:

$$\beta = \frac{\int_S \; v^2 dS}{\bar{v}^2 A} \tag{53}$$

For most turbulent flows $\beta \sim 1.01 - 1.04$, and the value $\beta = 1$ is considered. For a laminar flow in a pipe, $\beta = 4/3$.

The uniform, steady, two-dimensional flow with free surface of a real fluid: As an application of the three fundamental principles of the physics applied to hydraulics, we will analyse the flow in an infinitely wide inclined channel in which the permanent and uniform flow with a free surface exists. The problem is sketched in Fig. 10. The bottom of the channel coincides with the x axis which is inclined an angle θ with respect to the horizontal line. The flow depth is H and the cross sectional mean velocity is U. There is not heat transfer between the fluid system and the environment and the flow is turbulent.

The first thing that needs to be done when working with the equations derived with the integral approach, is to choose the control volume. In this case V_C corresponds to the volume defined by the segmented line. The application of the continuity equation, energy equation and momentum theorem must be done in this volume.

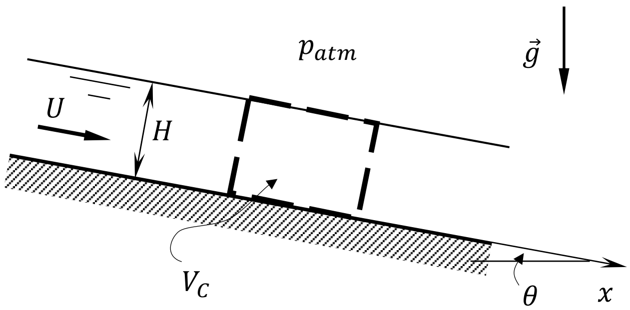


Fig. 10.- Steady, uniform, 2-D free surface

Continuity equation: As the flow is 2-D, we work with both the discharge and the area of the flow section per unit width, i.e., $q = Q/b$ and $a = A/b$. Calling (1) to the entrance section and (2) to the exit section, and considering steady flow, the application of Eq. 1 to the control volume reduces to $q_1 = q_2$. Uniformity of the flow means that the flow depth in both sections is the same, from where $a_1 = a_2 = H$. Thus, from Eq. 2: $v_1 = v_2 = U$.

Energy equation: The control volume corresponds to a streamtube, as shown in Fig. 11. As the flow is turbulent we assume $\alpha = 1$. Application of Eq. 45:

$$B_2 = B_1 - \Lambda \tag{45}$$

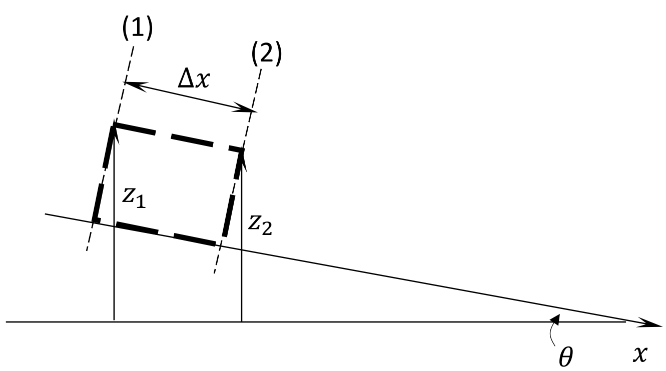


Fig. 11.- Application of the energy equation to the control volume

As the flow is uniform, the streamlines are parallels, and the sum $z + p/(\rho g)$ remains constant in any cross section. That means that the sum can be evaluated at any z

of the section. For simplicity, we choose to evaluate it at the free surface, because at that location the pressure is known (p_{atm}). Thus, Eq. 45 becomes:

$$z_2 + \frac{p_{atm}}{\rho g} + \frac{U^2}{2g} = z_1 + \frac{p_{atm}}{\rho g} + \frac{U^2}{2g} - \Lambda \tag{54}$$

Thus:

$$\Lambda = z_1 - z_2 \tag{55}$$

From the geometry:

$$\sin\theta = -\frac{z_1 - z_2}{\Delta x} \tag{55}$$

Using Eq. 46 and $\Lambda = J\Delta x$, we obtain:

$$J = \sin\theta \tag{56}$$

The result above obtained indicates that for steady, uniform open-channel flows the gradient of the energy line is equal to the slope of the channel. Or, equivalently, the energy line is parallel to the bottom of the channel (and parallel to the free surface).

We can relate the flow velocity with the slope of the channel using Eq. 47. It is easy to show that for this flow, $R_H = H$. Thus, we have:

$$\sin\theta = f\frac{1}{4H}\frac{U^2}{2g} \tag{57}$$

The problem is not solved yet because we have not said anything about the friction factor. We will get some relationships for it later.

Momentum theorem: In order to apply Newton's second law to the control volume of Fig. 10, we have to recognize that Eq. 52 is a vectorial equation, and it should be applied to each component of the coordinate system. As we already have defined the control volume, the first step in our analysis is to identify the forces and momentum fluxes acting in V_C. To do this, we will use Fig. 12.

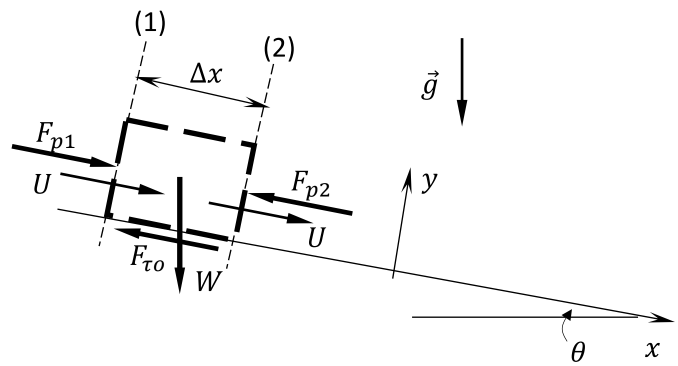


Fig. 12.- Application of the momentum theorem to the control volume

The y-component of the momentum equation is not relevant in this analysis. It only says that the component of the weight in the y direction is equilibrated with the reaction normal to the bottom of the channel. Let's analyse the x-component of the momentum equation. Eq. 52 gives:

$$\rho Q(\beta_2 v_{2x} - \beta_1 v_{1x}) = F_x \tag{58}$$

F_x is x-component the resultant of the forces acting in the control volume. We can identify the following forces: F_p , forces due to the fluid pressure; W, weight of the fluid contained in the control volume; and $F_{\tau o}$, force due to the friction between the fluid and the bottom. Thus:

$$F_x = F_{p1} - F_{p2} + W \sin\theta - F_{\tau 0} \tag{59}$$

As $v_{1x} = v_{2x} = U$, there is not variation of momentum, and Eq. 58 becomes simply:

$$F_{p1} - F_{p2} + W \sin\theta - F_{\tau 0} = 0 \tag{60}$$

We can evaluate the force due to the pressure as $F_p = p_G A$, where p_G is the fluid pressure at the centre of gravity of the surface with area A. As the flow is uniform, A and p_G are the same in sections (1) and (2), and the pressure does not contribute in the equilibrium of forces given by Eq. 60. This indicates that for the steady uniform flow in a channel, there is an equilibrium between the force that generates the motion (the component of the weight in the flow direction) and the force that opposes to the motion (due the friction at the wall):

$$W \sin\theta = F_{\tau 0} \tag{61}$$

The weight (per unit width) is given by:

$$W = \rho g H \Delta x \tag{62}$$

The force (per unit width) due to friction is:

$$F_{\tau 0} = \tau_0 \Delta x \tag{63}$$

In the above equation, τ_0 is the shear stress acting on the bottom of the channel. Eqs. 61, 62 and 63 give:

$$\tau_0 = \rho g H \sin\theta \tag{64}$$

Using Eq. 56, we can write:

$$\tau_0 = \rho g H J \tag{65}$$

Although Eq. 65 was obtained for steady and uniform flow, the same result is assumed to hold for steady and gradually varied flow.

Combining Eqs. 57 and 64:

$$\tau_0 = \frac{1}{8}\rho f U^2 \tag{66}$$

A "shear velocity", u_*, is defined as:

$$u_* = \sqrt{\frac{\tau_0}{\rho}} \tag{67}$$

From where:

$$\frac{u_*}{U} = \sqrt{\frac{f}{8}} \tag{68}$$

It is important to have in mind that u_* is not a flow velocity. It is only a way to write the friction stress acting on the wall, τ_0. The term $\sqrt{\tau_0/\rho}$ appears many times in the analysis of fluid flows. As it has dimensions of velocity, it is called "friction velocity".

Shear stress distribution in a steady, uniform, two-dimensional flow with free surface of a real fluid: We obtained the shear stress acting on the bottom of the channel (Eq. 64). As there is not shear applied on the free surface, there the shear stress is zero. The question is: how does the shear stress changes from zero at the free surface to $\tau_0 = \rho g H \sin\theta$ on the bottom?. To answer this question, we can applied the momentum theorem to the control volume of Fig.13.

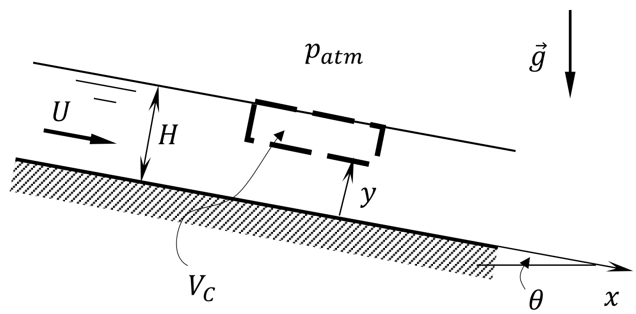


Fig. 13.- Control volume chosen to determine the variation of the shear stress

The forces acting in C_V are depicted in Fig. 14. Essentially, they are the same than those shown in Fig. 12, but for a control volume that does not reach the bottom of the channel.

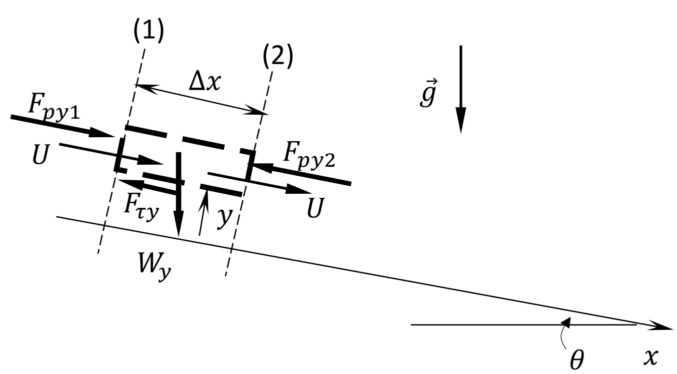


Fig. 14.- Forces acting in the control volume

F_{py} is the force due to the fluid pressure acting on the surfaces comprised between H and y, W_y is the weight of the fluid contained in the control volume, and $F_{\tau y}$ the force due to friction at a distance y from the bottom. With the same arguments that we obtained Eq. 61, we get:

$$W_y \sin\theta = F_{\tau y} \tag{69}$$

The weight of fluid and force due to shear stress are given by:

$$W_y \sin\theta = \rho g (H - y)\Delta x \tag{70}$$

$$F_{\tau y} = \tau_y \Delta x \tag{71}$$

From where we obtain:

$$\tau_y = \rho g (H - y) \sin\theta \tag{72}$$

Eq. 72 indicates that the shear stress varies linearly with depth, from zero on the free surface ($y = H$) to $\tau_y = \rho g H \sin\theta$ on the bottom ($y = 0$), as sketched in Fig. 15. Dividing Eq. 72 by Eq.64:

$$\frac{\tau_y}{\tau_0} = \left(1 - \frac{y}{H}\right) \tag{73}$$

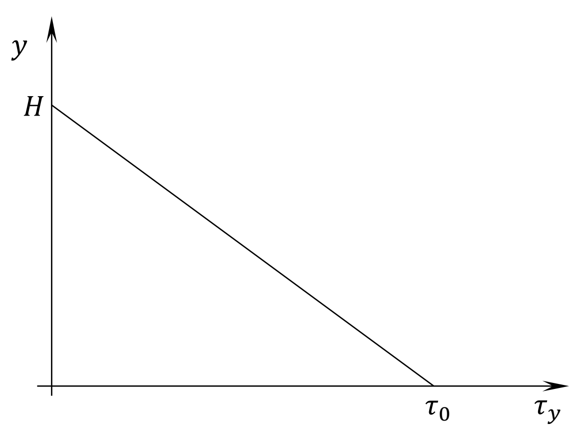


Fig. 15.- Shear stress distribution

BASIC EQUATIONS OF FLUID MECHANICS OBTAINED FROM THE DIFFERENTIAL APPROACH

As we have seen, the equations obtained using the integral approach do not provide information about the flow characteristics in a point of the flow domain, but it considers mean velocities, and total forces acting on the control volume or its surfaces. In order to take into account the variation of the velocity in a given section, the coefficients α and β appears, but from their definition, the velocity distribution is needed to compute them analytically. Now we will derive the equations of continuity and momentum considering an infinitely small control volume, which we will take to the limit that it becomes a point.

CONTINUITY EQUATION

Let's analyse the flow of mass through an element of volume $dV = dxdydz$ immersed in the flow, as shown in Fig. 16. The velocity field is $\vec{v} = (u, v, w)$. Through this imaginary volume, the flow (represented by the streamlines in the figure) passes transporting mass of fluid. Conservation of mass indicates that the variation of mass per unit time inside dV is equal to the mass rate that enters the volume, less that the mass rate that exits from it. This statement can be written as:

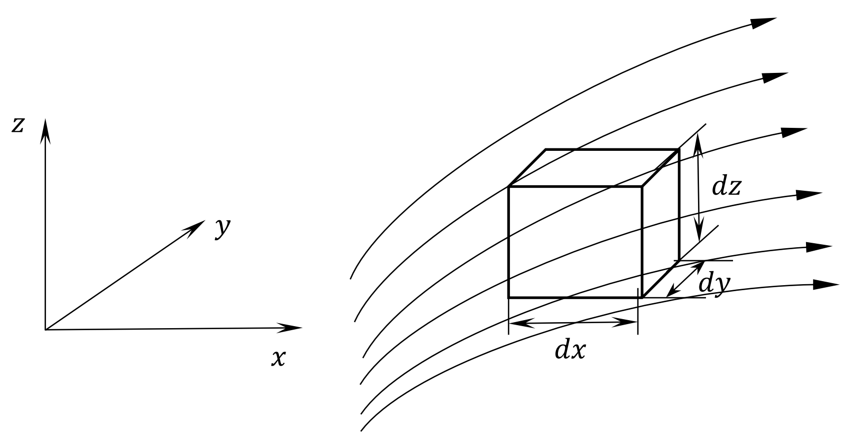


Fig. 16.- Imaginary element of volume in the flow domain

$$\frac{\partial m}{\partial t} = G_i - G_o \tag{74}$$

where m is the mass of fluid contained in dV, given by

$$dm = \rho dV \tag{75}$$

G_i is the mass rate that enters dV, and G_o is the mass rate that exits from the volume. The fluid can enter (and exit) through any of the six surfaces that define the volume. For the flow in the x direction, we have a surface located at x with normal $(-\hat{\imath})$, and other located at $x + dx$, with normal $(+\hat{\imath})$.The same happens for the other coordinates. Thus, we can decompose the mass that enters to dV in three terms, and write:

$$G_i = G_x|_x + G_y\big|_y + G_z|_z \tag{76}$$

where the symbol $|_x$ stands for "evaluated at x", and so on for the other directions. Similarly, the mass that exits the volume can be written as:

$$G_o = G_x|_{x+dx} + G_y\big|_{y+dy} + G_z|_{z+dz} \tag{77}$$

Replacing in Eq. 74:

$$\frac{\partial}{\partial t}(\rho dV) = G_x|_x + G_y\big|_y + G_z|_z - \left(G_x|_{x+dx} + G_y\big|_{y+dy} + G_z|_{z+dz}\right) \tag{78}$$

The terms evaluated at $x + dx$, $y + dy$ and in $z + dz$ can be related to those evaluated at x, y and in z by means of a Taylor's expansion. For example, the term $G_x|_{x+dx}$ is expanded as:

$$G_x|_{x+dx} = G_x|_x + \left.\frac{\partial G_x}{\partial x}\right|_x dx + \frac{1}{2}\left.\frac{\partial^2 G_x}{\partial x^2}\right|_x (dx)^2 + \cdots \tag{79}$$

As dx is infinitely small, the term $(dx)^2$ and higher powers of dx can be neglected. Thus, the net flow of mass along the x direction is:

$$G_x|_x - G_x|_{x+dx} = -\left.\frac{\partial G_x}{\partial x}\right|_x dx \tag{80}$$

The mass rate is defined as the discharge times the density. The discharge in the x direction is udA_x, where dA_x is the element of area of the surface with normal $\hat{\imath}$, $dA_x = dydz$. Thus, Eq. 80 becomes:

$$-\left.\frac{\partial G_x}{\partial x}\right|_x dx = -\left.\frac{\partial}{\partial x}(\rho u)\right|_x dxdydz \tag{81}$$

In the same way:

$$-\frac{\partial G_y}{\partial y}\bigg|_y dy = -\frac{\partial}{\partial y}(\rho v)\bigg|_y dxdydz \qquad -\frac{\partial G_z}{\partial z}\bigg|_z dz = -\frac{\partial}{\partial z}(\rho w)\bigg|_z dxdydz \tag{82}$$

and Eq. 78 becomes:

$$\frac{\partial \rho}{\partial t} dxdydz = -\left(\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right) dxdydz \tag{83}$$

Thus, the continuity equation in the differential approach is written as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{84}$$

or, in vectorial form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{85}$$

For an incompressible fluid, Eq. 85 is greatly reduced to:

$$\nabla \cdot \vec{v} = 0 \tag{86}$$

Or, equivalently:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{87}$$

Eq. 87 (or 86) is the equivalent to Eq.1 of the integral approach.

MOMENTUM EQUATION

Before applying the Newton's second law to the elementary volume of Fig. 16, it is important to remember that two kinds of forces can be identified in the element of fluid: forces that depend on the amount of matter and that apply to the centre of gravity of the element of fluid (for example, weight). They are called "body forces", and notated as $\vec{F}_B$. The second kind of forces act on the surfaces that define the volume of fluid (like pressure and shear stresses), and "surface forces", and written as $\vec{F}_S$. Thus, Newton's second law can be written as:

$$\frac{d}{dt}(m\vec{v}) = \vec{F}_S + \vec{F}_B \tag{88}$$

As the total mass of the system remains constant, Eq. 88 evolves into:

$$\rho dV \frac{d\vec{v}}{dt} = \vec{F}_S + \vec{F}_B \tag{89}$$

The total or material derivative of the velocity can be expressed as:

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z} \tag{91}$$

The body force depends of the amount of matter, i.e., it is proportional to the mass of fluid in the dV. As the mass is given by ρdV, we can express the body force in terms of a body per unit mass, $\vec{f}_B$:

$$\vec{F}_B = \rho \vec{f}_B dV \tag{92}$$

The most common body force is that due to gravity. In this case, $\vec{f}_B = \vec{g}$, and Eq. 92 becomes:

$$\vec{F}_B = \rho \vec{g} dV \tag{93}$$

The analysis of surface forces requires to evaluate the forces acting on the six surfaces that define the element of fluid volume, as shown in Fig. 17. There are three surface forces acting on each surface. The surface stresses are represented as τ_{ij}, where i indicates the direction of the vector normal to the surface and j the direction of the force (where i and j can take the values $\hat{\imath}$, $\hat{\jmath}$ or $\hat{k}$). Thus, the net force in the direction j is the result of the stresses of the surfaces located at x, with normal $(-\hat{\imath})$; at $x + dx$, with normal $(+\hat{\imath})$; at y, with normal $(-\hat{\jmath})$; at $y + dy$, with normal $(+\hat{\jmath})$; and at z, with normal $\left(-\hat{k}\right)$; at $z + dz$, with normal $\left(+\hat{k}\right)$. Obviously, the direction j can be $\hat{\imath}$, $\hat{\jmath}$ or $\hat{k}$.

The net surface force has components in the three directions:

$$\vec{F}_S = F_{Sx}\hat{\imath} + F_{Sy}\hat{\jmath} + F_{Sz}\hat{k} \tag{94}$$

As it was said, six surfaces contribute to each component of Eq. 94. For the component in the x direction we have:

$$F_{Sx} = F_{Sxx}|_x + F_{Sxx}|_{x+dx} + F_{Syx}\big|_y + F_{Syx}\big|_{y+dy} + F_{Szx}|_z + F_{Szx}|_{z+dz} \qquad (95)$$

In term of the stresses, any surface force can be written as $F_{Sij} = \tau_{ij} d\vec{S}_j$, where $d\vec{S}_j$ is the surface on which the stress is acting, with a vector normal to the surface in the j direction. Thus, Eq. 95 is given by:

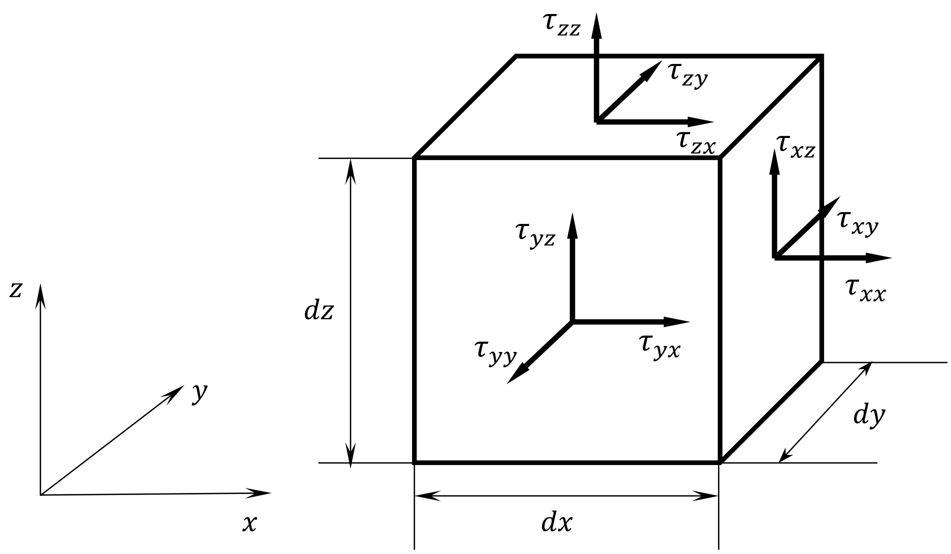


Fig. 17.- Surface forces acting on the three visible surfaces of the element of fluid volume

$$\begin{aligned} F_{Sx} = {} & \tau_{xx}|_x(-\hat{\imath})dS_x + \tau_{xx}|_{x+dx}(\hat{\imath})dS_x + \tau_{yx}\big|_y(-\hat{\jmath})dS_y + \tau_{yx}\big|_{y+dy}(\hat{\jmath})dy \\ & + \tau_{zx}|_z(-\hat{k})dS_z + \tau_{zx}|_{z+dz}(\hat{k})dS_z \end{aligned} \qquad (96)$$

The surfaces on which the stresses are acting correspond to:

$$dS_x = dydz \qquad dS_y = dxdz \qquad dS_z = dxdy \qquad (97)$$

Expanding in Taylor's series and neglecting the terms of second order and higher, we have:

$$\begin{aligned}
\tau_{xx}\big|_{x+dx} &= \tau_{xx}\big|_{x} + \frac{\partial \tau_{xx}}{\partial x}\bigg|_{x} dx + \cdots \\
\tau_{yx}\big|_{y+dy} &= \tau_{yx}\big|_{y} + \frac{\partial \tau_{yx}}{\partial y}\bigg|_{y} dy + \cdots \\
\tau_{zx}\big|_{z+dz} &= \tau_{zx}\big|_{z} + \frac{\partial \tau_{zx}}{\partial z}\bigg|_{z} dz + \cdots
\end{aligned} \tag{98}$$

Replacing Eqs. 97 and 98 in Eq. 96, we get:

$$F_{Sx} = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdydz \tag{99}$$

Following the same procedure for the components along y and z of $\vec{F}_S$, we obtain:

$$F_{Sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) dxdydz \tag{100}$$

$$F_{Sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) dxdydz \tag{101}$$

Replacing Eqs. 91, 93, 99, 100 and 101 in Eq. 94, Newton's second law is written as:

$$\begin{gathered}
\rho\left(\frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z}\right) = \rho\vec{g} + \\
\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\hat{\imath} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)\hat{k}
\end{gathered} \tag{102}$$

We can recognize that τ_{xx}, τ_{yx}, τ_{zx}, τ_{xy}, … are the elements of matrix array. Actually, they are a tensor, and τ_{ij} corresponds to the stress tensor:

$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix} \tag{103}$$

Eq. 102 is greatly simplified when it is written in vectorial form:

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = \rho\vec{g} + \nabla\cdot\tau_{ij} \tag{104}$$

As Eq. 104 is vectorial, we can write three equations, one for each component.

Component x:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z} \tag{105}$$

Component y:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z} \tag{106}$$

Component z:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z} \tag{107}$$

In the above equations, $\vec{g}=\left(g_x,g_y,g_z\right)$ was used.

Eq. 104 (or Eqs. 105 to 107) are the Cauchy's equations. It has to be noted that up to now, we have not imposed explicitly at what kind of matter is applied (the only restriction is that it has to be continuous, in a gravitational field). It can be applied as much as solids as to fluids. We define the kind of matter through the so-called "constitutive relationships". They are relationships between the stresses τ_{ij} and the deformation or deformation rate of the matter. We will restrict the analysis to Newtonian fluids (for example fluids like water, air, oil, etc.). We will not perform the derivation of the constitutive relationships for Newtonian fluids but give the final result, obtained by Stokes in 1845. Basically, the assumptions made by Stokes to derive the equation of motion of Newtonian fluids are the following:

1. The fluid is a continuum
2. If there is not motion, the equations of hydrostatic should be recovered
3. There is, at the most, a linear relationship between the stresses and the angular deformation rate of an element of fluid
4. The fluid is isotropic, i.e., the constitutive relationship is independent of the direction or coordinate system.

The second assumption indicates that if there is no motion, the angular deformation rates should be zero and the normal stresses acting on the element of fluid should reduce to the pressure. This means that

$$\tau_{ij}=-p\delta_{ij} \qquad \text{if } \vec{v}=0 \tag{108}$$

δ_{ij} is the Kronecker's delta, defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{109}$$

The sign ($-$) in Eq. 108 results from the fact that the pressure points towards the element of fluid. The final result from Stokes analysis is:

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda\nabla\cdot\vec{v} \tag{110}$$

(As before, i and j indicate the component along x, y or z)

μ is the dynamic viscosity and λ is the second coefficient of viscosity.

Stokes defined the *mechanical pressure*, $\bar{p}$, as the negative average of the normal stresses:

$$\bar{p} = -\frac{\tau_{xx} + \tau_{yy} + \tau_{zz}}{3} \tag{111}$$

From Eq. 110, we get

$$\begin{aligned} \tau_{xx} &= -p + 2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\vec{v} \\ \tau_{yy} &= -p + 2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\vec{v} \\ \tau_{zz} &= -p + 2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\vec{v} \end{aligned} \tag{112}$$

Replacing the last equations in Eq. 111:

$$\bar{p} = p - \left(\frac{2}{3}\mu + \lambda\right)\nabla\cdot\vec{v} \tag{113}$$

Note that Eq. 113 gives an interesting result: the mechanical pressure, $\bar{p}$, is not equal to the thermodynamic pressure, p. An interesting issue is raised with the second coefficient of viscosity. In his work, Stokes assumed that $(2\mu/3 + \lambda) = 0$, meaning that $\lambda < 0$. However, some measurements indicate that $\lambda > 0$. Nevertheless, the value of λ should not bother us because for incompressible fluids $\nabla\cdot\vec{v} = 0$ (Eq. 86), and in this case $\bar{p} = p$. Also, for uncompressible fluids, Eq. 110 is reduced to:

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{114}$$

Thus, the nine components of the tensor τ_{ij} (Eq. 103) are:

$$\tau_{xx} = -p + 2\mu \frac{\partial u}{\partial x} \quad (115)$$

$$\tau_{yy} = -p + 2\mu \frac{\partial v}{\partial y} \quad (116)$$

$$\tau_{zz} = -p + 2\mu \frac{\partial w}{\partial z} \quad (117)$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad (118)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \quad (119)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \quad (120)$$

The terms containing μ are the viscous stresses. Introducing Eqs. 115 to 120 in Eqs.105 to 107 and using Eq. 87, we obtain the momentum equations for an incompressible Newtonian fluid with constant viscosity in the gravitational field:

Component x:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x \quad (121)$$

Component y:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y \quad (122)$$

Component z:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g_z \quad (123)$$

Eqs. 121, 122 and 123 can be written in a more compact form using vectorial notation:

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla p + \mu\nabla^2\vec{v} + \rho\vec{g} \quad (124)$$

Eqs 121 to 123 (or Eq. 124) are called the Navier-Stokes equations. The steps given in this note to obtain the equations are based in the work of George Gabriel Stokes "On the Theories of Internal Friction of Fluids in Motion", published in 1845, but in 1822 Claude-Louis Marie Henri Navier presented the article "Memoire sur les lois du mouvement des fluides" (published in 1823), where he got the same sets of Eqs. 121 to 123, but instead of the viscosity μ multiplying the second derivatives of the velocity, he had a coefficient $-\varepsilon$ that arose from the resistance generated by the slip of adjacent layers of fluid molecules (Navier, 1823, p. 416). Although the equations obtained by Navier have the right form, he was not able to link the origin of the molecular forces with viscosity.

The unknowns of a problem of fluid motion are four: the three components of the velocity field (u, v, w) and the pressure (p). In general, they are functions of the space and time: (x, y, z, t).Conceptually, the problem is already solved, because we have a system of four partial differential equations, with their corresponding boundary and initial conditions. Three equations corresponds to the Navier-Stokes equations (Eqs. 121, 122 and 123) and the fourth is the continuity equation (Eq. 87). However, the set differential equations is highly complex, due to the nonlinear terms $(u\,\partial u/\partial x\,, v\,\partial u/\partial y\,, \dots)$ of the momentum equations. Thus, analytical solutions are restricted to a few simple cases. The existence of solutions of the Navier-Stokes equations and if they are unique (for the general case) has not been proved yet, and it is one of the problem of the millennium. There is a prize of US$ 1 million to be awarded to whom can prove that the solution exists and it is unique. More information about the prize can be found in the website www.claymath.org/millennium-problems/millennium-prize-problems. As an anecdote, we can mention that in 2013, Mukhtarbay Otelbaev, from the Institute of Mathematics and Mathematical Modelling, Kazakhastan, published the paper (in Russian) "Existence of a strong solution of the Navier-Stokes equation" in the Mathematical Journal (Almaty), Vol 13, No. 4, pp. 5-104, where through 101 pages he would have solved the problem. However, about a month later, Otelbaev recognized that he made a mistake in one of the steps of his deduction (Moskvitch, 2014). As a recognition of the importance of the Navier-Stokes, one student of the Department of Mining Engineering of the University of Chile tattooed the equations in his arm, as shown in Fig. 18xx. As he works with slurries that behave as non-Newtonian fluid, the constitutive relationship differs from Eqs. 115 to 120 and he wrote only τ. For non-Newtonian fluids, the expression for τ can be very complex and the student will need his other arm to tattoo them!

In the development of the Navier-Stokes equations we have not imposed any restriction regarding the flow regime, and they are valid as much for laminar as for turbulent flows. However, analytical solutions are obtained only for laminar flows. The approach to turbulent flows by means of the Navier-Stokes equations is done numerically, and it has open a complete field of research y fluid mechanics, the computational fluid dynamics (CFD).

Fig. 18xx.- Tattoo in the arm of one student of the Mining Engineering Department of the University of Chile.

The steady, uniform 2-D laminar flow with free surface over an inclined plane. To fix ideas, we will solve now the problem that we already analysed using the integral approach. A fluid of density ρ and dynamic viscosity μ flows over a plane inclined an angle θ with a flow depth H due to the action of gravity. The flow is steady, uniform and laminar. The problem is to determine the velocity and pressure distributions.

We choose the coordinate system indicated in Fig. 18. As the flow is 2-D, we can omit the z component of the momentum equation and drop w and z derivatives. Thus the

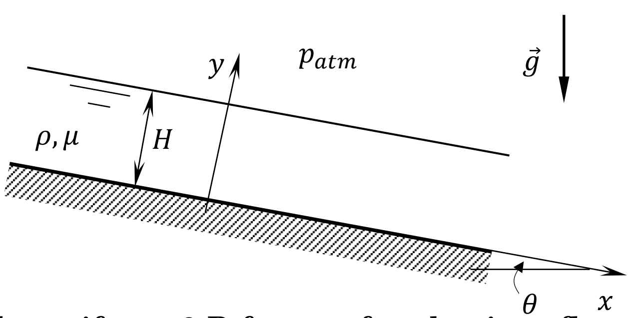


Fig. 18.- Steady, uniform, 2-D free surface laminar flow over an inclined plane

equation of continuity and Navier-Stokes are reduced to:

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{125}$$

As the flow is uniform, the streamlines are parallel to the x axis (the free surface is parallel to the bottom). Thus, there is not component of the velocity in the y direction. Therefore:

$$v = 0 \qquad 0 \le y \le H \tag{126}$$

and

$$\frac{\partial u}{\partial x} = 0 \tag{127}$$

Navier-Stokes equation in the x direction:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho g_x \tag{128}$$

Navier-Stokes equation in the y direction:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g_y \tag{129}$$

Eqs. 128 and 129 are greatly simplified with the conditions of the problem and the result obtained from the continuity equation (Eqs. 126 and 127)

The condition of steady state means that the partial derivatives with respect to t are zero. This condition and Eqs. 126 and 127 leads to:

Navier-Stokes equation in the x direction:

$$0 = -\frac{\partial p}{\partial x} + \rho g_x + \mu\frac{\partial^2 u}{\partial y^2} \tag{130}$$

Navier-Stokes equation in the y direction:

$$0 = -\frac{\partial p}{\partial y} + \rho g_y \tag{131}$$

The last equation indicates that the pressure distribution varies linearly with y, i.e., we have a hydrostatic pressure distribution. Integrating Eq. 131:

$$p = \rho g_y y + C_1(x) \tag{132}$$

$\partial(\text{Eq}.132)/\partial x$ and replacing it in Eq. 130:

$$0 = -\frac{\partial C_1}{\partial x} + \rho g_x + \mu \frac{\partial^2 u}{\partial y^2} \tag{133}$$

From Eq.127 we know that the third term of Eq. 133 does not depend on x. It means that we can take that term as a constant K if we integrate Eq. 133 with respect to x:

$$\frac{\partial C_1}{\partial x} = \rho g_x + K \tag{134}$$

$$C_1 = (\rho g_x + K)x + C \tag{135}$$

C is a pure constant because we already know that C_1 does not depend on y. Replacing C_1 in Eq. 132:

$$p = \rho g_y y + (\rho g_x + K)x + C \tag{136}$$

To determine C we have to apply the boundary condition for the pressure. At the free surface we have atmospheric pressure. Working with relative pressures, $p_{atm} = 0$ and the boundary condition becomes:

$$\text{For any } x \text{ at } y = H, \quad p = 0 \tag{137}$$

Thus:

$$C = -\rho g_y H - (\rho g_x + K)x \tag{138}$$

Replacing now the C in Eq. 136:

$$p = \rho g_y (y - H) \tag{139}$$

From Fig. 18 it is easy to see that

$$g_x = g \sin\theta \quad , \quad g_y = -g\cos\theta \tag{140}$$

Finally, the pressure distribution is:

$$p = \rho g \cos\theta\,(H - y) \tag{141}$$

Known the pressure, we can go back to Eq. 130. From Eq. 141, we have $\partial p/\partial x = 0$, thus:

$$\mu \frac{\partial^2 u}{\partial y^2} = -\rho g_x \tag{142}$$

Using the cinematic viscosity, $\nu = \mu/\rho$ and Eq. 140:

$$\frac{\partial^2 u}{\partial y^2} = -\frac{g}{\nu}\sin\theta \tag{143}$$

Integrating Eq. 143 twice with respect to y:

$$u = -\frac{g}{\nu}\sin\theta\frac{y^2}{2} + Ay + B \tag{144}$$

The constants A and B are determined from the boundary conditions. At the bottom we have the non-slip condition, i.e., the fluid velocity is the same that the bottom velocity. It means:

$$\text{For any } x \text{ at } y = 0, \quad u = 0 \tag{145}$$

The second boundary condition is on the free surface. As there is not shear stress applied on it, the condition is:

$$\text{For any } x \text{ at } y = H, \quad \tau_{yx} = 0 \tag{146}$$

That is to say:

$$\tau_{yx}\Big|_{y=H} = \mu\frac{\partial u}{\partial y}\Big|_{y=H} = 0 \tag{147}$$

The following values are obtained:

$$A = \frac{g}{\nu} \sin\theta \, H$$
$$B = 0 \tag{148}$$

Finally, the velocity distribution is given by:

$$u = \frac{g}{\nu} \sin\theta \left(Hy - \frac{y^2}{2} \right) \tag{149}$$

From Eq. 149 we see that the velocity distribution is parabolic, with a maximum value at the free surface equal to $u = g \sin\theta \, H^2/(2\nu)$. We can also compute the average velocity, $\bar{u} = U$:

$$U = \frac{1}{H} \int_0^H u \, dy$$
$$U = \frac{1}{3} \frac{g}{\nu} \sin\theta \, H^2 \tag{150}$$

We can write Eq. 150 in dimensionless form multiplying both sides by $U/(gH)$:

$$\frac{U^2}{gH} = \frac{1}{3} \frac{UH}{\nu} \sin\theta \tag{151}$$

We recognize in the left hand side of Eq. 151 a Reynolds number based on the flow depth:

$$Re_H = \frac{UH}{\nu} \tag{152}$$

In the left hand side there is a dimensionless number that appears when gravity forces are important. It is the square of Froude number, which is defined as:

$$Fr = \frac{U}{\sqrt{gH}} \tag{153}$$

The equation that relates the average velocity of the flow with its depth is called in hydraulics “resistance law”, which in this case can be written as:

$$Fr^2 = \frac{1}{3} Re_H \sin\theta \tag{154}$$

As we know the velocity distribution, we can also compute the coefficients α and β introduced in the integral approach (Eqs. 38 and 53), resulting $\alpha = 54/35 \approx 1.543$ and $\beta = 18/15 = 1.2$.

We can also compute the shear stress distribution and its value at the bottom, that we called τ_0:

$$\tau_{yx} = \mu \frac{\partial u}{\partial y}$$
$$\tau_{yx} = \rho g \sin\theta (H - y) \tag{155}$$

At the bottom ($y = 0$):

$$\tau_0 = \rho g H \sin\theta \tag{156}$$

Note that the relationships given by Eqs. 155 and 156 are the same than those obtained with the application of the momentum theorem (Eqs. 72 and 64). Something that we cannot obtain from the integral approach is the resistance relationship given by Eq. 154. The equivalent equation in the integral approach is the Darcy-Weisbach equation (Eq. 47 that evolved into Eq. 57). However, this equation requires to know the friction factor f, that should be determined from other way (theoretically, numerically or experimentally). Using Eqs. 57 and 154 we can obtain the friction factor for a steady, uniform, 2-D laminar flow in a channel:

$$\sin\theta = f \frac{U^2}{8gH} = 3 \frac{Fr^2}{Re_H} \tag{157}$$

From where:

$$f = \frac{24}{Re_H} \tag{158}$$

Do not confuse the expression given by Eq. 158 with the expression for a cylindrical pipe, $f = 64/Re$ (and shown graphically in Moody's diagram). Replacing in Re the diameter D by $4R_H$ we do not obtain Eq. 158 but $f = 16/Re_H$, which is wrong. The factor 24 has been confirmed experimentally (Chow, 1988).

2.- REYNOLDS' EQUATIONS FOR THE TURBULENCE

Turbulent flows are unsteady flows, in the sense that at any point of the flow domain, the flow properties $(\vec{v}, p)$ are fluctuating in time. This is shown in Fig.2.1, where the three components of the velocity measured with an acoustic Dopper velocimeter (ADV) are presented. It is easily seen the fluctuating characteristic of the data. It is rather difficult to use the data as it is presented in the figure. Osborne Reynolds in 1895 proposed that in the turbulent flow velocity and pressure could be decomposed into two terms, called by him *mean-mean-motion* and *relative-mean-motion* (Reynolds, 1895). The peer reviewers of the paper were G.G. Stokes and H. Lamb. In the first response that they sent to the editor (Lord Rayleigh), Stokes acknowledged that he did not understand the work, and Lamb indicated that much of the paper was obscure (Jackson and Launder, 2007). Currently, we call those terms *mean* (or *average*) and *fluctuation*. But, mean or average of what?

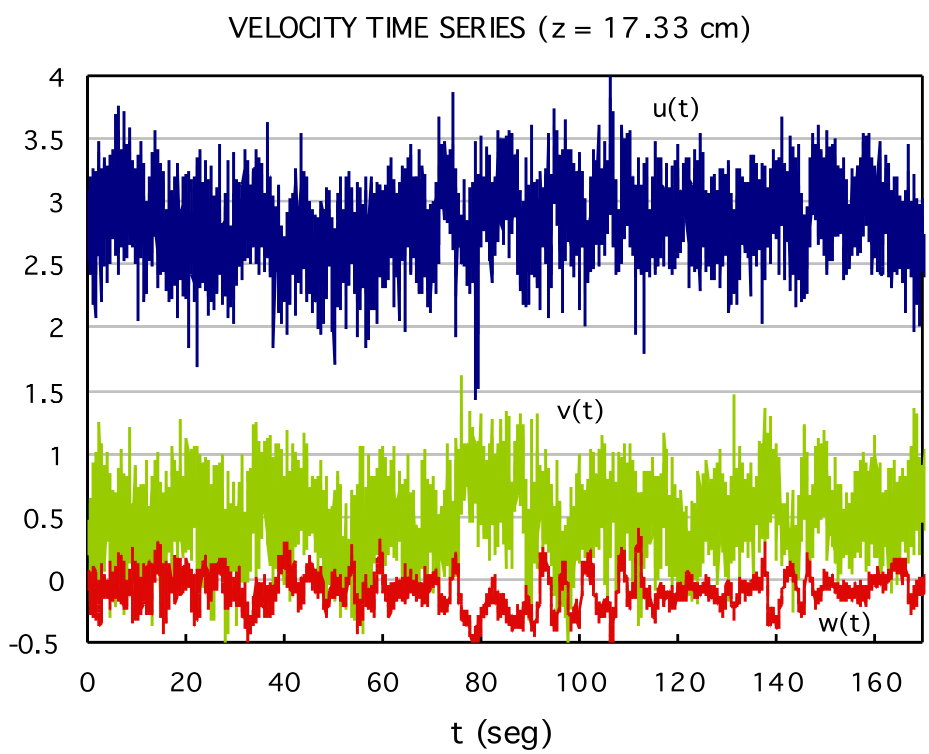


Fig. 2.1.- Record of the three components of the velocity measured at one location in a turbulent flow in an open channel

The simplest (or easier to understand) is the *temporal mean* or *temporal average*. Considering, for example, the record along the time t of the u component of the velocity at a given location $\vec{x}$ of the flow domain, the temporal average is defined as:

$$\bar{u}(\vec{x}) = \frac{1}{T}\int_{t_0}^{t_0+T} u(\vec{x},t)\ dt \tag{2.1}$$

WARNING! Unfortunately, we are using the same notation for to different averages. We used the overbar $\bar{\ }$ previously to denote the average velocity in a cross section ($\bar{v} = Q/A$), and now we are using the overbar to denote an average on time. Be careful!

If the average given by Eq. 2.1 does not depend on t_0, the process is called *statistically stationary*.

According to the Reynolds decomposition, the velocity $u(\vec{x},t)$ can be split in two components: its temporal average $\bar{u}(\vec{x})$ and a fluctuation $u'(\vec{x},t)$, such that:

$$u(\vec{x},t) = \bar{u}(\vec{x}) + u'(\vec{x},t) \tag{2.2}$$

The two components presented in Eq. 2.2 are shown in Fig. 2.2. It is easy to show that:

$$\bar{\bar{u}} = \bar{u} \qquad \overline{u'} = 0 \tag{2.3}$$

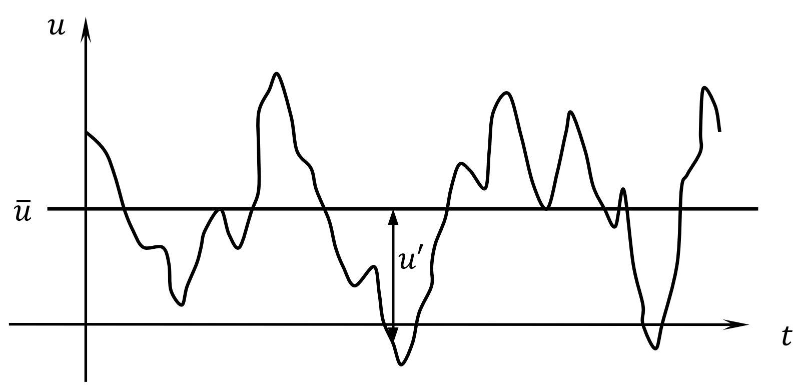


Fig. 2.2.- Reynolds decomposition

Obviously, the same decomposition is valid for the other components of the velocity and pressure:

$$v = \bar{v} + v' \quad w = \bar{w} + w' \quad p = \bar{p} + p' \tag{2.4}$$

There is another kind of average, the *ensemble average*, which is more appropriate for the analysis. In this case, the experiment is repeated many times (N), each repetition is called a realization. Thus, for realization 1, we have a record $u_1(\vec{x},t)$, for realization 2 we have a record $u_2(\vec{x},t)$, and so on. An average of the N realizations for each $(\vec{x},t)$ gives the ensemble average $\langle u(\vec{x},t)\rangle$, defined as:

$$\langle u(\vec{x},t)\rangle = \frac{1}{N}\sum_{i=1}^{N} u_i(\vec{x},t) \tag{2.2}$$

Note that the result of the ensemble average at a location $\vec{x}$, is not a value as in the temporal average, but a function of time. This can be seen in Fig. 2.3. For statistical processes that are *ergodic and stationaries*, ensemble and temporal averages are equal:

$$\langle u\rangle = \bar{u}_i \tag{2.3}$$

In the analysis that follows we will consider that the experimental data obtained in a turbulent flow behave statistically as an ergodic process *strictly* stationary (i.e., not only the averages are the same but also any other statistical property of the flow).

It is easy to show the following properties for any variable b that is a function of time

$$\overline{\frac{\partial b}{\partial x_i}} = \frac{\partial \bar{b}}{\partial x_i} \tag{2.4}$$

$$\overline{\int b\, dx_i} = \int \bar{b}\, dx_i \tag{2.5}$$

Replacing the Reynolds-decomposed variables (Eqs. 2.2 y 2.4) into continuity equation (Eq. 87):

$$\frac{\partial(\bar{u}+u')}{\partial x} + \frac{\partial(\bar{v}+v')}{\partial y} + \frac{\partial(\bar{w}+w')}{\partial z} = 0 \tag{2.6}$$

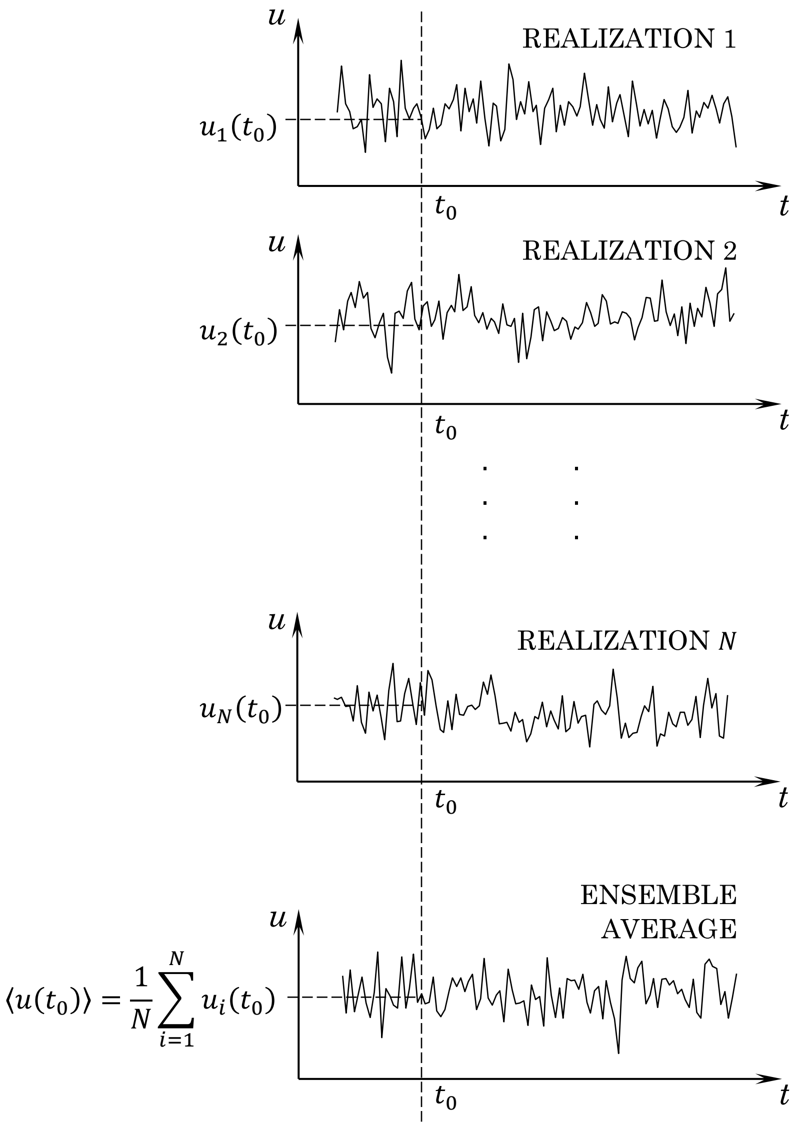


Fig. 2.3.- Ensemble average

Averaging Eq. 2.6:

$$\overline{\frac{\partial(\bar{u}+u')}{\partial x}+\frac{\partial(\bar{v}+v')}{\partial y}+\frac{\partial(\bar{w}+w')}{\partial z}}=0 \tag{2.7}$$

$$\overline{\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}+\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}}=0 \tag{2.8}$$

$$\frac{\partial \bar{\bar{u}}}{\partial x}+\frac{\partial \bar{\bar{v}}}{\partial y}+\frac{\partial \bar{\bar{w}}}{\partial z}+\frac{\partial \overline{u'}}{\partial x}+\frac{\partial \overline{v'}}{\partial y}+\frac{\partial \overline{w'}}{\partial z}=0 \tag{2.9}$$

Using Eqs. 2.3:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0 \tag{2.10}$$

Substracting Eq. 2.10 to Eq. 2.6:

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0 \tag{2.11}$$

Eqs. 2.10 and 2.11 indicate that the averaged velocities and their fluctuations satisfy continuity.

Before repeating the same analysis with the Navier-Stokes equations, we will modify slightly the equations. We will work first with the x component of the momentum equation. Let's multiply the continuity equation by u:

$$u\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial y}+u\frac{\partial w}{\partial z}=0 \tag{2.12}$$

Multiplying Eq. 2.12 by ρ and adding it to the x component of the Navier-Stokes equation (Eq. 121):

$$\begin{aligned}\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+u\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial y}+u\frac{\partial w}{\partial z}\right)\\=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x\end{aligned} \tag{2.13}$$

We can recognize some product derivatives in the first parenthesis:

$$\begin{gathered} u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = 2u\frac{\partial u}{\partial x} = \frac{\partial u^2}{\partial x} \\ v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} = \frac{\partial uv}{\partial y} \\ w\frac{\partial u}{\partial z} + u\frac{\partial w}{\partial z} = \frac{\partial uw}{\partial z} \end{gathered} \tag{2.14}$$

Thus, the x component of the Navier-Stokes is rewritten as

$$\rho\left(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x \tag{2.13}$$

Now, we will replace the Reynolds decomposed variables in Eq. 2.13: and take the average. The resulting equation is:

$$\begin{gathered} \rho\left(\frac{\partial\overline{(\bar{u}+u')}}{\partial t} + \frac{\partial\overline{(\bar{u}+u')^2}}{\partial x} + \frac{\partial\overline{(\bar{u}+u')(\bar{v}+v')}}{\partial y} + \frac{\partial\overline{(\bar{u}+u')(\bar{w}+w')}}{\partial z}\right) \\ = -\frac{\partial\overline{(\bar{p}+p')}}{\partial x} + \mu\nabla^2\overline{(\bar{u}+u')} + \rho g_x \end{gathered} \tag{2.14}$$

The linear terms in Eq. 2.14 are easily decomposed into their average and fluctuating parts:

$$\frac{\partial\overline{(\bar{u}+u')}}{\partial t} = \frac{\partial\overline{\bar{u}}}{\partial t} + \frac{\partial\overline{u'}}{\partial t} = 0 \tag{2.15}$$

$$\frac{\partial\overline{(\bar{p}+p')}}{\partial x} = \frac{\partial\overline{\bar{p}}}{\partial x} + \frac{\partial\overline{p'}}{\partial x} = \frac{\partial\bar{p}}{\partial x} \tag{2.16}$$

$$\nabla^2\overline{(\bar{u}+u')} = \nabla^2\overline{\bar{u}} + \nabla^2\overline{u'} = \nabla^2\bar{u} \tag{2.17}$$

Let's analyse now the non-linear terms:

$$\frac{\partial\overline{(\bar{u}+u')^2}}{\partial x} = \frac{\partial\overline{\left(\bar{u}^2 + 2\bar{u}u' + u'^2\right)}}{\partial x} = \frac{\partial\overline{\bar{u}^2}}{\partial x} + \frac{\partial\overline{2\bar{u}u'}}{\partial x} + \frac{\partial\overline{\overline{u'^2}}}{\partial x} = \frac{\partial\bar{u}^2}{\partial x} + \frac{\partial\overline{u'^2}}{\partial x} \tag{2.18}$$

$$\frac{\partial \overline{(\bar{u}+u')(\bar{v}+v')}}{\partial y} = \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \overline{u'v'}}{\partial y} \tag{2.19}$$

$$\frac{\partial \overline{(\bar{u}+u')(\bar{w}+w')}}{\partial z} = \frac{\partial \bar{u}\bar{w}}{\partial z} + \frac{\partial \overline{u'w'}}{\partial z} \tag{2.20}$$

Thus, the averaged x component of the Reynolds equation becomes:

$$\rho\left(\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \bar{u}\bar{w}}{\partial z} + \frac{\partial \overline{u'w'}}{\partial z}\right) = -\frac{\partial \bar{p}}{\partial x} + \mu\nabla^2\bar{u} + \rho g_x \tag{2.21}$$

The right hand side of Eq. 2.21 can also be written as

$$\begin{aligned} &\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \bar{u}\bar{w}}{\partial z} + \frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} \\ &= 2\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial \bar{v}}{\partial y} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{u}\frac{\partial \bar{w}}{\partial z} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} \\ &+ \frac{\partial \overline{u'w'}}{\partial z} \end{aligned} \tag{2.22}$$

$$\begin{aligned} &\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \bar{u}\bar{w}}{\partial z} + \frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} \\ &= \bar{u}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right) + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} \\ &+ \frac{\partial \overline{u'w'}}{\partial z} \end{aligned} \tag{2.23}$$

Using Eq. 2.10, the term in parenthesis can be eliminated and (2.23) written as:

$$\begin{aligned} &\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \bar{u}\bar{w}}{\partial z} + \frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} \\ &= \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} \end{aligned} \tag{2.24}$$

Eq. 2.21 becomes:

$$\rho\left(\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z}+\frac{\partial\overline{u'^2}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right)=-\frac{\partial\bar{p}}{\partial x}+\mu\nabla^2\bar{u}+\rho g_x \tag{2.25}$$

Remember that Eq. 2.21 is nothing else than Newton's second law applied to an incompressible Newtonian fluid and expressed in terms of temporal mean values. In order to interpret the meaning of Eq. 2.21, we should remember the equation of Newton's second law: $d(m\vec{v})/dt=\vec{F}$. As the mass is preserved, we can write $md(\vec{v})/dt=\vec{F}$. For simplicity, let's consider the x component:

$$m\frac{du}{dt}=F_x \tag{2.26}$$

Of course, Newton's second law is not applied to fluids in the form of Eq. 2.26, but as Eq. 51 (integral approach) or Eq. 124 (differential approach), but it is used in this explanation for the sake of clarity. The problem with Eq. 2.26 (and the reason why in fluids is expressed in a different way, is to define the mass m in a flow. Anyway, we can divide Eq. 2.26 by a volume and work with forces per unit volume, $F_{Vx}\equiv F_x/V$:

$$\rho\frac{du}{dt}=F_{Vx} \tag{2.27}$$

We can identify F_{Vx} in Eq. 124 as the resulting of the force due to pressure, viscosity, and gravity. We would like to work with a temporal mean acceleration, and write Eq. 2.27 as:

$$\rho\frac{d\bar{u}}{dt}=\bar{F}_x \tag{2.28}$$

We can define an acceleration based on the temporal mean quantities. For a statistically stationary flow this acceleration is:

$$\frac{d\bar{u}}{dt}=\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z} \tag{2.29}$$

Thus, introducing Eq. 2.29 into Eq. 2.25:

$$\rho\left(\frac{d\bar{u}}{dt}+\frac{\partial\overline{u'^2}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right)=-\frac{\partial\bar{p}}{\partial x}+\mu\nabla^2\bar{u}+\rho g_x \tag{2.30}$$

We can see that Eq. 2.30 is very similar to Eq. 2.28. The only problem is the terms due to the fluctuations in the parenthesis. In order to have only the variation of momentum in the left hand side of Eq. 2.30, we take the terms due to the fluctuations towards the right hand side of the equation:

$$\rho\frac{d\bar{u}}{dt} = -\frac{\partial\bar{p}}{\partial x} + \mu\nabla^2\bar{u} + \rho g_x - \rho\left(\frac{\partial\overline{u'^2}}{\partial x} + \frac{\partial\overline{u'v'}}{\partial y} + \frac{\partial\overline{u'w'}}{\partial z}\right) \tag{2.31}$$

To pass the terms due to the fluctuations from one side of the equal sign to the other side is much more than an algebraic step. It changes the interpretation that we can give to the average of products of the fluctuations. As they are now in the right side of the equation, we can interpret them as forces arising from the turbulence. Thus, the terms $-\rho\overline{u'^2}$, $-\rho\overline{u'v'}$ and $-\rho\overline{u'w'}$ are called *Reynold's apparent stresses* or, simply, *Reynold's stresses* , or *turbulent stresses*. Let's work with the terms associated to the viscous stresses $\mu\nabla^2\bar{u}$:

$$\nabla^2\bar{u} = \frac{\partial^2\bar{u}}{\partial x^2} + \frac{\partial^2\bar{u}}{\partial y^2} + \frac{\partial^2\bar{u}}{\partial z^2} \tag{2.32}$$

As the continuity equation is $\nabla\cdot\vec{v} = 0$, we can take its derivative with respect to x and add to Eq. 2.32, and it will not change, i.e.:

$$\nabla^2\bar{u} = \nabla^2\bar{u} + \frac{\partial}{\partial x}\nabla\cdot\vec{\bar{v}} \tag{2.33}$$

$$\nabla^2\bar{u} = \frac{\partial^2\bar{u}}{\partial x^2} + \frac{\partial^2\bar{u}}{\partial y^2} + \frac{\partial^2\bar{u}}{\partial z^2} + \frac{\partial^2\bar{u}}{\partial x\partial x} + \frac{\partial^2\bar{v}}{\partial x\partial y} + \frac{\partial^2\bar{w}}{\partial x\partial z} \tag{2.34}$$

$$\nabla^2\bar{u} = \frac{\partial}{\partial x}\frac{\partial\bar{u}}{\partial x} + \frac{\partial}{\partial y}\frac{\partial\bar{u}}{\partial y} + \frac{\partial}{\partial z}\frac{\partial\bar{u}}{\partial z} + \frac{\partial}{\partial x}\frac{\partial\bar{u}}{\partial x} + \frac{\partial}{\partial y}\frac{\partial\bar{v}}{\partial x} + \frac{\partial}{\partial z}\frac{\partial\bar{w}}{\partial x} \tag{2.35}$$

$$\nabla^2\bar{u} = \frac{\partial}{\partial x}\left(2\frac{\partial\bar{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial x}\right) \tag{2.36}$$

Multiplying Eq. 2.36 by μ:

$$\mu\nabla^2\bar{u} = \frac{\partial}{\partial x}\left(2\mu\frac{\partial\bar{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial x}\right)\right) \tag{2.37}$$

We identify the terms in parenthesis as some of the viscous stress that appear in Eq. 114. Denoting with the sub-index V to denote "viscous", we have:

$$\tau_{Vxx} = 2\mu\frac{\partial\bar{u}}{\partial x}\,, \quad \tau_{Vyx} = \mu\left(\frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x}\right), \quad \tau_{Vzx} = \mu\left(\frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial x}\right) \tag{2.38}$$

We can write now:

$$\mu\nabla^2\bar{u} = \frac{\partial\tau_{Vxx}}{\partial x} + \frac{\partial\tau_{Vyx}}{\partial y} + \frac{\partial\tau_{Vzx}}{\partial z} \tag{2.39}$$

Replacing in Eq. 2.31:

$$\rho\frac{d\bar{u}}{dt} = -\frac{\partial\bar{p}}{\partial x} + \frac{\partial\tau_{Vxx}}{\partial x} + \frac{\partial\tau_{Vyx}}{\partial y} + \frac{\partial\tau_{Vzx}}{\partial z} + \rho g_x - \rho\left(\frac{\partial\overline{u'^2}}{\partial x} + \frac{\partial\overline{u'v'}}{\partial y} + \frac{\partial\overline{u'w'}}{\partial z}\right) \tag{2.40}$$

$$\rho\frac{d\bar{u}}{dt} = -\frac{\partial\bar{p}}{\partial x} + \frac{\partial}{\partial x}\left(\tau_{Vxx} - \rho\overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\tau_{Vyx} - \rho\overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\tau_{Vzx} - \rho\overline{u'w'}\right) + \rho g_x \tag{2.41}$$

Using the sub-index T to denote the turbulent or Reynolds stresses:

$$\tau_{Txx} = -\rho\overline{u'^2}\,, \quad \tau_{Tyx} = -\rho\overline{u'v'}\,, \quad \tau_{Tzx} = -\rho\overline{u'w'} \tag{2.42}$$

We define the total stress as the sum of the viscous and the turbulent one:

$$T_{xx} = \tau_{Vxx} + \tau_{Txx}\,, \quad T_{yx} = \tau_{Vyx} + \tau_{Tyx}\,, \quad T_{zx} = \tau_{Vzx} + \tau_{Tzx} \tag{2.43}$$

In general:

$$T_{ij} = \tau_{Vij} + \tau_{Tij} \tag{2.44}$$

Where the viscous stress (associated to the temporal mean velocities) is given by:

$$\tau_{Vij} = \mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{2.45}$$

And the turbulent stress is:

$$\tau_{Tij} = -\rho\overline{u_i'u_j'} \tag{2.46}$$

Thus, the momentum equation in the $x$direction for a stationary turbulent flow, in terms of the temporal mean values is given by:

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}\right) = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} + \rho g_x \tag{2.47}$$

In the same way, we can obtain the y and z components:

$$\rho\left(\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \bar{w}\frac{\partial \bar{v}}{\partial z}\right) = -\frac{\partial \bar{p}}{\partial y} + \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} + \rho g_y \tag{2.48}$$

$$\rho\left(\bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{v}\frac{\partial \bar{w}}{\partial y} + \bar{w}\frac{\partial \bar{w}}{\partial z}\right) = -\frac{\partial \bar{p}}{\partial z} + \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} + \rho g_z \tag{2.49}$$

Eqs. 2.47, 2.48 and 2.49 are the Reynolds equations for turbulent flows, and they constitute an important advance in the study and analysis of the turbulence. In his paper, Reynolds also derived and discussed the equation for the turbulent kinetic energy.

Although we have had some progress in the analysis of turbulent flows, we are not in conditions to solve any problem yet. We have a system formed by 4 partial differential equations (with their boundary conditions): Eqs. 2.10, 2.47, 2.48 and 2.49. But we have 10 unknowns: $\bar{u}, \bar{v}, \bar{w}, \bar{p}, \overline{u'^2}$, $\overline{v'^2}, \overline{w'^2}, \overline{u'v'}$, $\overline{u'w'}$, $\overline{v'w'}$. We cannot solve any problem of turbulence using the equations derived with the Reynolds decomposition approach if we do not know relations for $-\rho\overline{u_i'u_j'}$. This is the so-called "closure problem of turbulence". Basically, the problem is how to model the turbulent stresses. As there is not a theory based only on the first principles of the physics, all the available models necessarily require some experimental data. We will present two closures of the problem, both of them widely used in engineering. They are the Boussinesq's eddy viscosity model and Prandtl mixing length theory.

Boussinesq closure of the turbulence: Eddy viscosity

Boussinesq published in 1877 a compilation of his research on water flows. (The book can be read in the site http://gallica.bnf.fr/ark:/12148/bpt6k56673076, but it cannot be downloaded). In Eqs. 12 of his book, Boussinesq presents the stresses in a similar form (but with slightly different notation) than those given in Eq. 114 in tensorial notation, or Eqs. 115 to 120 (see Fig. 2.4). The big difference is that in the later equations, appears μ, and in Boussinesq's equations there is a coefficient ε. But before to discuss the meaning and value of ε, it is interesting to note that Boussinesq was solving a problem that arose almost 20 years later, with Reynolds. It is interesting to note that Boussinesq already considered temporal averages, but his mistake was to assume that the velocity fluctuations were not correlated, i.e., $\overline{u_i' u_j'} = 0$ (using our notation). In this way, he lost additional components to the stresses. However, it was known that the equations of Navier-Stokes provided results in agreement with the experiments for conduits with small flow area, but it failed for larger conduits. In the latter case, additional effects appear in the flow, with the final effect that the viscosity seems to be larger (in our language we would say that the Navier-Stokes were in agreement with measurements in laminar flows, but other effects should be taking into account when dealing with turbulent flows. Details are in the first 50 pages of Boussinesq's book. For open channel flows he gives:

$$\varepsilon = \rho g A h u_0 \tag{2.50}$$

Eq. 2.50 corresponds to Boussinesq's Eq. 13. In an open channel flow, h is the flow depth and u_0 is the "velocity at the wall". A is a coefficient that depends on the wall roughness varies little with h and u_0. Note that in order that Eq. 2.50 be dimensionally homogeneous, the dimensions of A must be T^2L^{-1} ($time^2/length$).

$$(12)\ \begin{cases} N_1 = -p + 2\varepsilon \dfrac{du}{dx}, \quad N_2 = -p + 2\varepsilon \dfrac{dv}{dy}, \quad N_3 = -p + 2\varepsilon \dfrac{dw}{dz}, \\ T_1 = \varepsilon\left(\dfrac{dv}{dz} + \dfrac{dw}{dy}\right), \quad T_2 = \varepsilon\left(\dfrac{dw}{dx} + \dfrac{du}{dz}\right), \quad T_3 = \varepsilon\left(\dfrac{du}{dy} + \dfrac{dv}{dx}\right). \end{cases}$$

§ III. EXPRESSION APPROCHÉE DU COEFFICIENT ε DES FROTTEMENTS INTÉRIEURS.

Fig. 2.4.- Equations for the stresses presented by Boussinesq in 1877 in his *Essai sur la théorie des eaux courantes*. ε is the eddy viscosity.

In our current language, ε is named *eddy* viscosity, or *turbulent viscosity*. Thus, according to Boussinesq, the turbulent stresses τ_{Tij}, can be computed in the same way than the viscous stresses, using the eddy viscosity ε instead of μ, the molecular dynamic viscosity. The turbulent stresses become:

$$\tau_{Txx} = 2\varepsilon \frac{\partial \bar{u}}{\partial x} \tag{2.51}$$

$$\tau_{Tyy} = 2\varepsilon \frac{\partial \bar{v}}{\partial y} \tag{2.52}$$

$$\tau_{Tzz} = 2\varepsilon \frac{\partial \bar{w}}{\partial z} \tag{2.53}$$

$$\tau_{Txy} = \tau_{Tyx} = \varepsilon \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) \tag{2.54}$$

$$\tau_{Txz} = \tau_{Tzx} = \varepsilon \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right) \tag{2.55}$$

$$\tau_{Tyz} = \tau_{Tzy} = \varepsilon \left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y}\right) \tag{2.60}$$

There is a strong difference between μ and ε: *the dynamic viscosity μ is a property of the fluid* and *the eddy viscosity ε is a property of the flow*. Thus, if we are working with water at a given temperature, we can look for the value of the viscosity in any book and use it, independently if the flow is in a cylindrical or square pipe, a rectangular or trapezoidal channel. On the contrary, the eddy viscosity depends on the flow and the geometry and it is independent of the fluid (at least, for flows with high Reynolds numbers). As it is was said in the Introduction, one of the features of the turbulent flows is they are highly efficient in the momentum transfer process, when compared with laminar flows. Thus, another characteristic of the eddy viscosity is that it is much larger (several orders of magnitude) than the dynamic viscosity:$\varepsilon \gg \mu$.

In analogy to the molecular kinematic viscosity $\nu = \mu/\rho$, a kinematic eddy viscosity is also used $\nu_T = \varepsilon/\rho$. Eddy viscosities for some flow configurations are presented in Table 2.1. Known the eddy viscosity, we can solve the problem of getting the velocity distribution in a turbulent flow. We have to solve the system of partial differential equations formed by continuity and Reynolds equations, in addition to an expression for the eddy viscosity.

WARNING! In the current technical literature ε is frequently used to design the *kinematic eddy viscosity* (i.e., $\nu_T = \varepsilon$) and not the dynamic turbulent viscosity, as Boussinesq designed in his original work and shown in Fig. 2.4..

As an example of Boussinesq's eddy viscosity approach to close Eqs. 2.47 to 2.49, we can solve the problem of the 2-D steady, uniform, turbulent flow on an inclined plane that we have been using as example along this notes (Fig. 18). Using the same arguments than in the laminar case, the equations of continuity (Eq. 2.10) and momentum (Reynolds equations, Eqs. 2.47 to 2.49) are reduced to:

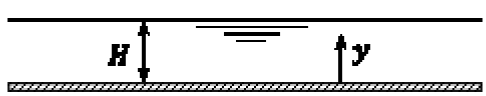
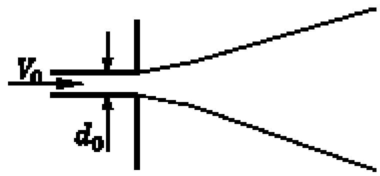
$$\frac{\partial \bar{u}}{\partial x} = 0 \tag{2.61}$$

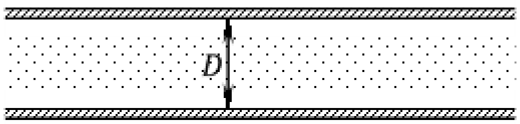
Reynolds equation in the x direction:

$$0 = \frac{\partial T_{yx}}{\partial y} + \rho g_x \tag{2.62}$$

Reynolds equation in the y direction:

Table 2.1.- EDDY VISCOSITY FOR SOME FLOW CONFIGURATIONS

<table>
<tr><th>FLOW CONFIGURATION</th><th>EDDY VISCOSITY</th></tr>
<tr><td>2-D Open channel flow

</td><td>$\nu_T = \kappa H u_* \eta(1-\eta)$
$\kappa = 0.4 \;\; ; \quad \eta = \dfrac{y}{H}$</td></tr>
<tr><td>Axisymetric jet

</td><td>$\nu_T = 0.013 V_0 d_0$</td></tr>
</table>

Central region of the flow in a pipe  	$\frac{\nu_T}{\nu} = \frac{C}{2} Re \sqrt{\frac{f}{8}}$ $C \approx 0.07 \quad ; \quad Re = \frac{UD}{\nu}$

$$0 = -\frac{\partial \overline{p}}{\partial y} + \rho g_y \tag{2.63}$$

From where we obtain that the mean pressure distribution is hydrostatic. With the boundary condition $y = H$, $\overline{p} = 0$ (relative pressure), we obtain:

$$\overline{p} = \rho g \cos\theta \,(H - y) \tag{2.64}$$

Reynolds equation along x (Eq. 2.62) can be integrated once with respect to y:

$$T_{yx} = -\rho g_x y + C_1 \tag{2.65}$$

As there is not shear acting on the free surface, the boundary condition at $y = H$ is $T_{yx} = 0$. Thus:

$$C_1 = \rho g_x H \tag{2.66}$$

Using Eq. 2.43 and the value of C_1 in Eq. 2.65:

$$\tau_{Vyx} + \tau_{Tyx} = \rho g_x (H - y) \tag{2.67}$$

Replacing the shear stresses by Eqs. 118 and 2.54:

$$(\mu + \varepsilon)\frac{\partial \overline{u}}{\partial y} = \rho g_x (H - y) \tag{2.68}$$

Considering that the molecular viscosity μ is much smaller than the eddy viscosity ε:

$$\varepsilon \frac{\partial \bar{u}}{\partial y} \cong \rho g_x (H - y) \tag{2.69}$$

To drop μ from the momentum equation limits its use only to the region where turbulence dominates over viscosity. We will see later that even in highly turbulent flows bounded by smooth walls, in a small region near the wall molecular viscosity effects are important.

In order to integrate Eq. 2.69, we have to decide what value of ε will be used. We can work with the expression given by Boussinesq (Eq. 2.50) or that presented in Table 2.1.

Solution considering Boussinesq expression for ε:

Eq. 2.50 is $\varepsilon = \rho g A h u_0$, and Eq. 2.69 becomes:

$$\frac{\partial \bar{u}}{\partial y} = \frac{g_x}{\rho g A h u_0}(H - y) \tag{2.70}$$

Integrating:

$$\bar{u} = \frac{g_x}{g A h u_0}\left(Hy - \frac{1}{2}y^2\right) + C_2 \tag{2.71}$$

Although we still have to determine the integration constant C_2, the velocity distribution for a turbulent flow given by Eq. 2.71 is essentially the same than the distribution obtained for a laminar flow (Eq. 149) (parabolic distribution). This should not surprise us: we only changed the value of the constant that multiplies the derivative of the velocity. Determination of C_2 is conceptually more complicated. We cannot impose the condition that at $y = 0$, $\bar{u} = 0$ because very close to the bottom Eq. 2.69 is not valid. We should know the distance y_T from the wall where turbulence dominates, and the value of the velocity, u_T, at that location. Instead of doing that, we will follow Boussinesq ideas. According to him, in there is a velocity at the wall, u_0, which is rather large. In a footnote of his book (p. 51) he works a typical value of 1 m/s. Thus, according to Boussinesq, at $y = 0$, $\bar{u} = u_0$, and the resulting velocity distribution is:

$$\bar{u} = u_0 + \frac{\sin\theta}{A h u_0}\left(Hy - \frac{1}{2}y^2\right) \tag{2.72}$$

A comparison of the laminar and Boussinesq's turbulent velocity profiles are sketched in Fig. 2.4. A feature of the turbulent velocity profiles is that they are flatter than the laminar ones. In Boussinesq constant eddy viscosity model this is achieved by the term u_0, the fluid slip at the wall.

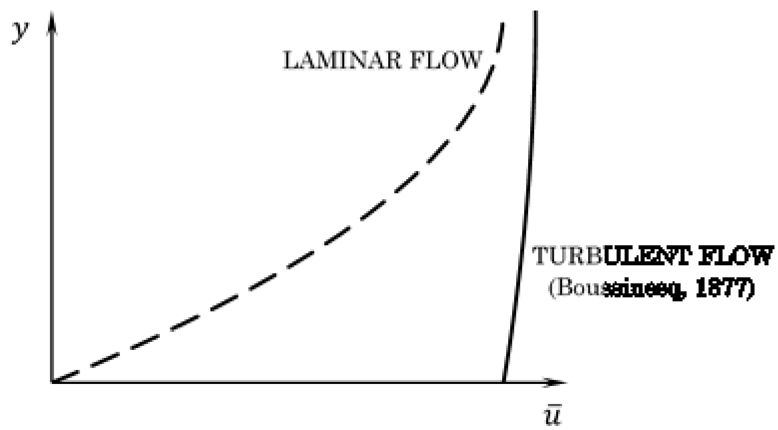


Fig. 2.4.- Comparison of the velocity distribution for laminar flow and turbulent flow according to the eddy viscosity model of Boussinesq (1877) that includes a velocity u_0 at the wall.

Solution considering the expression given in Table 2.1 for ε:

In this case, the eddy viscosity is given by:

$$\varepsilon = \rho\kappa u_* \frac{y}{H}(H - y) \tag{2.73}$$

The equation of motion gives:

$$\frac{\partial \bar{u}}{\partial y} = \frac{\rho g_x}{\varepsilon}(H - y) = \frac{g_x H}{\kappa u_* y} \tag{2.74}$$

Integrating:

$$\bar{u} = \frac{g_x H}{\kappa u_*}\ln(y) + C_3 \tag{2.75}$$

Again, we cannot use a boundary condition at $y = 0$ to determine the integration constant C_3. Mathematically, the logarithm in Eq. 2.75 blows up. Physically, the velocity distribution is valid only in the flow region where turbulence dominates (we have neglected μ). C_3 is determined by fitting Eq. 2.75 to experimental data. Using Eqs. 140 and 156 together to the definition of shear velocity (Eq. 67) $g_x H = gH \sin\theta = \tau_0/\rho = u_*^2$:

$$\bar{u} = \frac{u_*}{\kappa}\ln(y) + C_3 \tag{2.76}$$

The logarithmic velocity distribution given by Eq. 2.76 is sketched in Fig.2.5. It is observed that, for y smaller than a certain value, the velocity becomes negative and $\bar{u} \to -\infty$ when $y \to 0$. Obviously, the velocity distribution is valid only in the region where the flow is turbulent, i.e., for y larger than a specific distance from the bottom.

It is worth to remark that the velocity profiles obtained for turbulent flow relay on experimental data. The parabolic profile of Boussinesq requires to know experimentally the coefficient A and to have a method to determine u_0. The logarithmic profile require experiments to determine κ and C_3. The opposite happens with the velocity profile for laminar flow. The parabolic profile obtained from the Navier-Stokes equation and given by Eq. 149 only requires to know the properties of the fluid, the flow and the acceleration due to gravity.

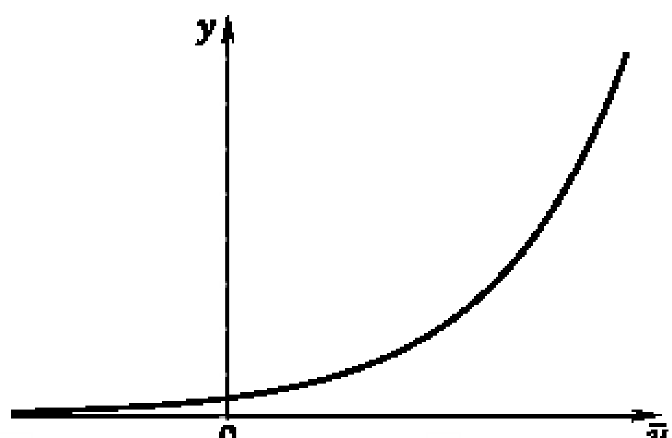


Fig. 2.5.- Logarithmic velocity profile. The velocity distribution is valid from a certain distance of the bottom, where the flow is turbulent.

Prandtl closure of the turbulence: Mixing length

Ludwig Prandtl is one of the great researchers in fluid mechanics of the XX century. His mixing length concept (or theory), is just one of his many contributions to the study of turbulence. It was proposed in 1925 (Prandtl, 1925).

In the mixing length theory, Prandtl assumes that portions of fluid can move due to the velocity fluctuations a distance l without losing its identity, basically preserving its momentum (per unit volume). To fix ideas, let's consider a 2-D flow with a mean velocity distribution as that sketched in Fig.2.6. In that figure, the continuous curve represents the

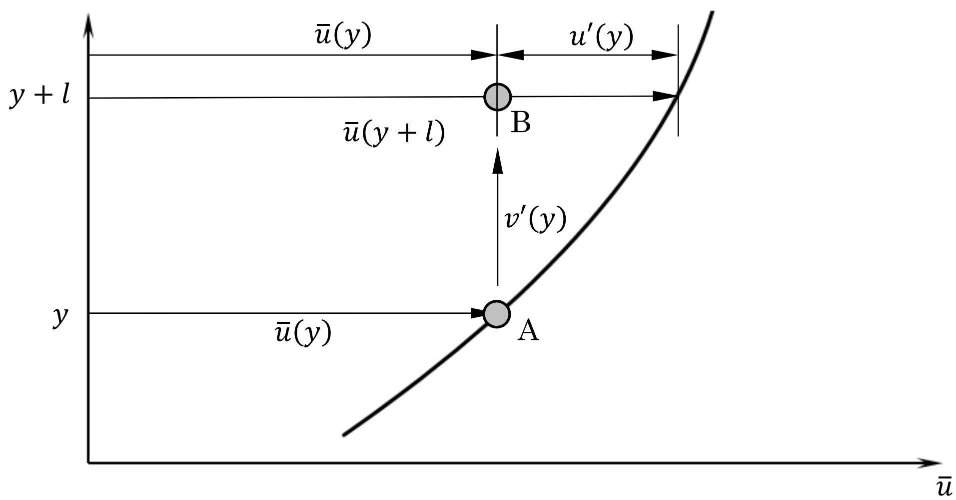


Fig. 2.6.- Concept of mixing length

mean velocity profile and the circles a mass of fluid. A mass of fluid that initially was at A with velocity $\bar{u}(y)$, due to the vertical fluctuation v' is transported to the location B, without losing its identity, i.e., preserving its momentum in the x direction (per unit volume) $\rho\bar{u}$. The distance of the displacement of the mass of fluid is l, named mixing length by Prandtl. As the mass of fluid did not change its properties during the displacement along the y axis, when it arrives to the location B has the velocity $\bar{u}(y)$, which is imposed at the level $(y + l)$. It is easy to see that the difference between this new velocity at $(y + l)$ and the mean velocity at this level, $\bar{u}(y + l)$, corresponds to the velocity fluctuation u':

$$u' = \bar{u}(y) - \bar{u}(y + l) \tag{2.77}$$

Expanding in Taylor's series and neglecting terms of second order and higher:

$$u' = \bar{u}(y) - \left(\bar{u}(y) + \frac{\partial \bar{u}}{\partial y} l + \cdots\right) \tag{2.78}$$

Thus:

$$u' = -l \frac{\partial \bar{u}}{\partial y} \tag{2.79}$$

Note that, for the mean velocity profile of Fig. 2.6, $\partial \bar{u}/\partial y > 0$, and $u' < 0$. The fluctuating velocity in the y direction was positive, $v' > 0$. This means that

$$u'v' < 0 \tag{2.80}$$

Experimental measurements show that $u'$and v' are well correlated, and they are of the same order of magnitude:

$$|u'| \sim |v'| \tag{2.81}$$

From the las two arguments (Eqs. 2.80 and 2.81), we can write:

$$v' \sim l \frac{\partial \bar{u}}{\partial y} \tag{2.82}$$

Thus, the turbulent shear stress $\tau_{Txy} = -\rho \overline{u'v'}$ (Eq. 2.42) can be written as:

$$\tau_{Txy} = \rho l^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 \tag{2.83}$$

From Eq. 2.83 we obtain that the eddy viscosity for this flow is given by:

$$\varepsilon = \rho l^2 \frac{\partial \bar{u}}{\partial y} \tag{2.84}$$

We can try to solve the problem of the permanent uniform flow over an inclined plane using Prandtl's expression for the turbulent shear stress (Eq. 2.83). We had obtained (Eq. 2.67):

$$\tau_{Vyx} + \tau_{Tyx} = \rho g_x (H - y) \tag{2.67}$$

Eqs. 2.38 and 2.83:

$$\mu \frac{\partial \bar{u}}{\partial y} + \rho l^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 = \rho g_x (H - y) \tag{2.85}$$

If we consider the region of the flow where turbulence dominates:

$$l^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 = g_x (H - y) \tag{2.86}$$

We cannot go further than Eq. 2.86 if we do not know an expression for the mixing length l. At this point we need to know l. The story is not simple, and involves the competition between Prandtl's research group in Gottingen and one of his former Ph.D. students, emigrated to the USA, Theodor von Kármán. In a paper presented in 1930 he assumes self-similarity of the velocity profiles and proposes (Kármán, 1930):

$$l = \kappa \frac{\frac{d\bar{u}}{dy}}{\frac{d^2\bar{u}}{dy^2}} \tag{2.87}$$

The proportionality coefficient κ is named von Karman coefficient and must be determined experimentally. Von Kármán recognized that Eq. 2.87 is not valid near the walls, because in those regions the turbulence is damped and viscosity needs to be taken into account. Von Karman applies his theory to the flow between two parallel plates separated a distance $2H$, as the centre-line is a symmetry axis, the problem is equivalent to the flow over an inclined plane. Replacing Eq. 2.87 in Eq. 2.86:

$$\kappa^2 \frac{\left(\frac{d\bar{u}}{dy}\right)^4}{\left(\frac{d^2\bar{u}}{dy^2}\right)^2} = g_x (H - y) \tag{2.88}$$

Making the change of variables $y' = H - y$ and calling $U' = d\bar{u}/dy = -\, d\bar{u}/dy'$ it is easy to perform the integration. Without going into details, and copying the result from von Karman's paper:

$$U' = \frac{d\bar{u}}{dy'} = \frac{\sqrt{g_x}}{2\kappa} \frac{1}{\sqrt{H} - \sqrt{y'}} \tag{2.89}$$

A word regarding the integration constant needs to be mentioned at this point. Following to von Karman, when we approach to the wall $d\bar{u}/dy$ becomes very large because μ is very small. Thus, he computes the integration constant evaluating at $y = 0\,, (y' = H)$

and imposing $U' \to \infty$. Eq. 2.89 is integrated again, with the boundary condition that the velocity is maximum (U_{max}) at t $y = H$, ($y' = H$). Using $u_* = \sqrt{gH\sin\theta}$ the velocity distribution is:

$$\bar{u} = U_{max} + \frac{u_*}{\kappa}\left[\ln\left(1 - \sqrt{1 - \frac{y}{H}}\right) + \sqrt{1 - \frac{y}{H}}\right] \tag{2.90}$$

Note that Eq. 2.90 fails at $y = 0$. This is correct because it is valid only in the region where the turbulent stresses dominate over the viscous ones.

Further, von Karman assumes that l should "quietly diminish to zero and the velocity distribution near to the wall becomes" (Eq. 25 in von Karman's paper):

$$\tau_{yx} - \mu\frac{\partial\bar{u}}{\partial y} = \rho(\kappa y)^2\left(\frac{\partial\bar{u}}{\partial y}\right)^2 \tag{2.91}$$

Von Karman recognizes that a Japanese researcher, Wada, got the equation in 1927, but the article was practically unknown because it was published in Japanese in a journal of naval architecture. According to von Karman, Wada applied Eq. 2.91 to the whole channel "which made the formulas a little too complicated". Von Karman's results are already contained in Wada's work "in an implied form". The right hand side term of Eq. 2.91 is the turbulent shear stress. A comparison with Eq. 2.83, gives the mixing length *in the turbulent region near the wall (or bottom)*:

$$l = \kappa y \tag{2.92}$$

In order to clarify the applicability of Eq. 2.92, consider the flow on a smooth wall, as sketched in Fig. 2.7. The existence of a "laminar layer" in contact with a smooth wall was known since the 1920's (von Karman, 1930). Due to the presence of the wall, the turbulent fluctuations are damped close to the wall, generating a region in which the viscous effects dominate over the turbulent ones ($\tau_{Vij} \ll \tau_{Tij}$). Now we call that region *viscous sublayer*. Far enough of the wall, turbulence dominates over viscosity and we have the *turbulent region*, where $\tau_{Tij} \ll \tau_{Vij}$. The passage from the viscous sublayer to the turbulent region is not abrupt and a *buffer layer* is identified in which viscous and turbulent effects are equally important ($\tau_{Vij} \sim \tau_{Tij}$). Further, the turbulent region can be divided in an *inner region* where there is a local energy equilibrium, and an *outer region*, which is dominated by large eddies which transport low momentum and high turbulent energy from the outer part of the inner region towards the upper part of the flow (Bradshaw, 1972).

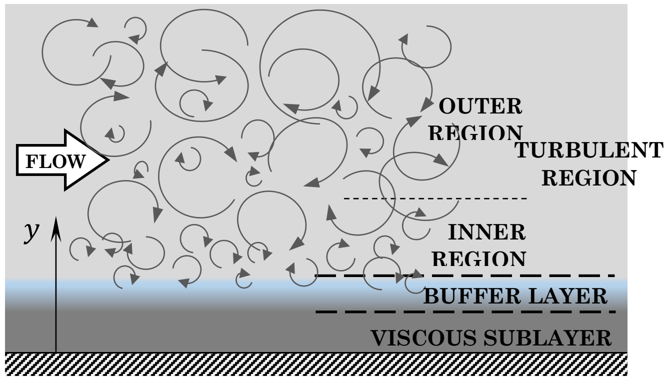


Fig. 2.7.- Regions of the turbulent flow over a smooth bottom.

The applicability of Eq. 2.92 is near the wall, but in the turbulent region. Using $l = \kappa y$ in Eq. 2.86, we have:

$$(\kappa y)^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 = g_x(H - y) \tag{2.93}$$

At this point if the analysis, we should mention that according to Schlichting (1979, p. 587) Eq. 2.92 was Prandtl's idea that was presented in a couple of papers in 1925 and 1926 (Since I do not read German, I cannot verify Schlichting's statement, but both papers do not contain an explicit equation similar to Eq. 2.92. It is possible that a linearly decreasing mixing length near the wall is mentioned in the text. Neither in von Kármán's article there is an explicit expression for the mixing length, but it is inferred from his Eq. 25, as indicated before).

Limiting the application of Eq. 2.93 to a region very near the wall, $y \ll H$, and recalling that $\tau_0 = \rho g H \sin\theta$, Eq. 2.93 is simplified to:

$$\kappa \frac{d\bar{u}}{dy} = \sqrt{\frac{\tau_0}{\rho}} \tag{2.94}$$

According to Schlichting (1979, p.588) to consider that the shear stress was constant and equal to that existing at the bottom was a "far-reaching assumption" introduced by Prandtl.

Integrating Eq. 2.94 and using the definition of shear velocity, the velocity distribution is:

$$\bar{u} = \frac{u_*}{\kappa} \ln(y) + C \tag{2.95}$$

Eq. 2.95 is named the logarithmic law for the velocity distribution or, simply, logarithmic law. The constantsκ and C needs to be determined from experimental data. We should not be surprised to see that we got the same result as that obtained with the eddy viscosity. The value of ε used to obtain the logarithmic profile given by Eq. 2.76 was obtained using the solution resulting from the mixing length (Eq. 2.95).

Although there is a discussion about the universality of von Karman's constant, it is usually taken as $\kappa = 0.4$. A dependency on the suspended sediments concentration exists. Determination of the constant C deserves a separated analysis, because it depends on the characteristics of the wall.

In addition to the mixing length given by Eq. 2.92, other have been proposed. For example, for the turbulent flow in a pipe of radius R, Nikuradse suggested (Schlichting, 1979):

$$\frac{l}{R} = 0.14 - 0{,}08\left(1 - \frac{y}{R}\right)^2 - 0{,}06\left(1 - \frac{y}{R}\right)^4 \qquad (2.96)$$

Buffer layer (region that assembles the viscous dominated layer near the wall with the turbulent region), van Driest proposed (Schlichting, 1979):

$$l = \kappa y\left[1 - \exp\left(\frac{1}{A}\frac{y u_*}{\nu}\right)\right] \quad , \quad A = 26 \qquad (2.97)$$

3.- VELOCITY DISTRIBUTIONS AND FRICTION FACTORS IN TURBULENT FLOWS

Smooth or rough wall?

Before presenting the velocity distribution, it is necessary to define the type of wall is the solid boundary of the conduit. Any surface observed with the proper magnifier will look rough, as sketched in Fig. 3.1. In fluid mechanics, the definition of smooth or rough wall does not depend only on the size of the roughness, but also in the thickness of the

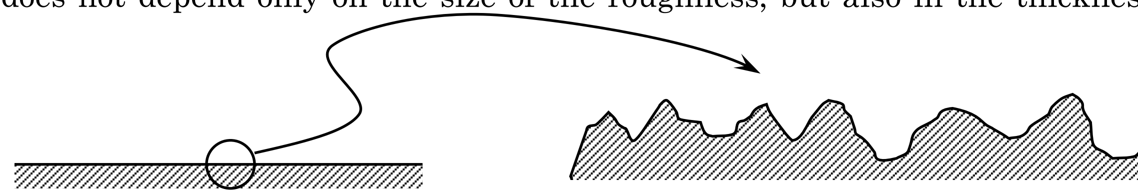


Fig. 3.1.- Roughness in a surface

viscous sublayer. The roughness size is commonly denoted by ε and the viscous sublayer thickness by δ_V. Thus, if the roughness is large enough to avoid the presence of a viscous sublayer ($\varepsilon > \delta_V$) the wall is hydraulically rough. On the contrary, if the viscous sublayer covers all the roughness ($\varepsilon < \delta_V$) the wall is hydraulically smooth. When the average size of the roughenss is around the viscous sublayer thickness ($\varepsilon \sim \delta_V$), the wall is in transition smooth-rough. To determine the limits that define the type of wall, we will follow Prandtl analysis to define the constant C of the logarithmic velocity distribution (Schlichting, 1979, p.589). As it is observed in Fig. 2.5, there is a distance y_0 at which the logarithmic profile is equal to cero ($\bar{u} = 0$). Taken it as a boundary condition, we obtain from Eq. 2.95:

$$\bar{u} = \frac{u_*}{\kappa}\left(\ln(y) - \ln(y_0)\right) \qquad (3.1)$$

The distance y_0 is of the order of magnitude of δ_V. Using dimensional analysis, we can get a dimensionless parameter involving y_0. The relevant variables near the bottom is the fluid properties (viscosity μ and density ρ), the shear stress acting on the wall (τ_0). Thus, we can expect a functional relationship of the form $y_0 = f(\mu, \rho, \tau_0)$. The number of variables is $n = 4$ and the number of fundamental dimensions is $r = 3$ (F,L,T). Applying the Buckingham Π theorem, we have only one dimensionless parameter ($n - r = 1$), given by:

$$\Pi = \frac{\tau_0^{1/2}\rho^{1/2}y_0}{\mu} \qquad (3.2)$$

Multiplying and dividing by $\rho^{1/2}$ and recalling that $u_* = \sqrt{\tau_0/\rho}$ and $\nu = \mu/\rho$ we get:

$$\Pi = \frac{u_* y_0}{\nu} \tag{3.3}$$

Thus we have a functional relationship $\Phi(\Pi) = 0$, that means that Π must be a constant. Calling β such a constant:

$$\frac{u_* y_0}{\nu} = \beta \tag{3.4}$$

Replacing y_0 from Eq. 3.4 into Eq. 3.1 and dividing by the shear velocity:

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\left(\ln\left(\frac{y u_*}{\nu}\right) - \ln(\beta)\right) \tag{3.5}$$

It is customary to define the *inner (or wall) variables*:

$$\bar{u}^+ \equiv \frac{\bar{u}}{u_*} \qquad , \qquad y^+ \equiv \frac{y u_*}{\nu} \tag{3.6}$$

Eq. 3.5 is re-written as:

$$\bar{u}^+ = \frac{1}{\kappa}\left(\ln(y^+) - \ln(\beta)\right) \tag{3.7}$$

The logarithmic profile written in dimensionless form as Eq. 3.7 will help us to define the viscous sublayer and buffer region thicknesses. However, before doing that, it will be useful to know the velocity distribution in the viscous sublayer.

Velocity distribution in the viscous sublayer

The characteristic of the viscous sublayer is that the viscous stresses are much larger than the turbulent ones, thus we can write $T_{yx} = \tau_{Vyx}$. In addition, as this is a thin layer, we can assume that the shear stress is not so different than its value on the bottom, τ_0. Thus, we can write:

$$\mu \frac{d\bar{u}}{dy} = \tau_0 \tag{3.8}$$

Integrating and imposing the boundary condition $y = 0$, $\bar{u} = 0$:

$$\mu \, \bar{u} = \tau_0 y \tag{3.9}$$

Dividing Eq. 3.9 by ρ:

$$\frac{\mu}{\rho}\,\bar{u} = \frac{\tau_0}{\rho}y \tag{3.10}$$

In terms of the inner variables:

$$\bar{u}^+ = y^+ \tag{3.11}$$

Thus, in the viscous sublayer the velocity varies linearly with the distance to the wall.

The viscous sublayer thickness

At this point, we have done a lot to determine the viscous sublayer thickness, but we need to rely on experimental data. The utility of having the velocity distribution in terms of dimensionless variables is that it is possible to compare data obtained under different flow conditions. Some experimental data is presented in Fig. 3.2, which was taken from White (1991, Fig. 6.11). In addition to the experimental data, the linear velocity profile form the viscous sublayer (Eq. 3.11), and the logarithmic profile form the turbulent region (Eq.3.5) are also plotted.

Obviously, the thickness of the viscous sublayer is defined by the distance from the wall where the experimental data departs from the linear profile (that appears as a curve in a semilogarithmic plot). This happens at $y^+ \approx 5$. Thus, the dimensionless viscous sublayer thickness, $\delta_V^+ = \delta_V u_*/\nu$ is taken as:

$$\delta_V^+ = 5 \tag{3.12}$$

The experimental data join the line defined by the logarithmic distribution around $y^+ \approx 70$. That means that for $y^+ > 70$ turbulence dominates over viscous effects. Finally the buffer layer, or region where the viscous effects are as important as the turbulent ones is comprised in $5 < y^+ < 70$. Often, the buffer region is neglected and $\delta_V^+ \sim 11$ is taken. Below that height the motion is considered laminar and above, fully turbulent.

For the flow in a smooth conduit, the three regions mentioned before are present. We are still talking about smooth walls but we have not given a precise definition regarding when a wall can be considered smooth. Following Prandtl (Schlichting, 1979) and von Karman (1930), a wall is hydraulically smooth when the roughness is completely contained within the viscous sublayer. A wall is hydraulically rough when the roughness size is large

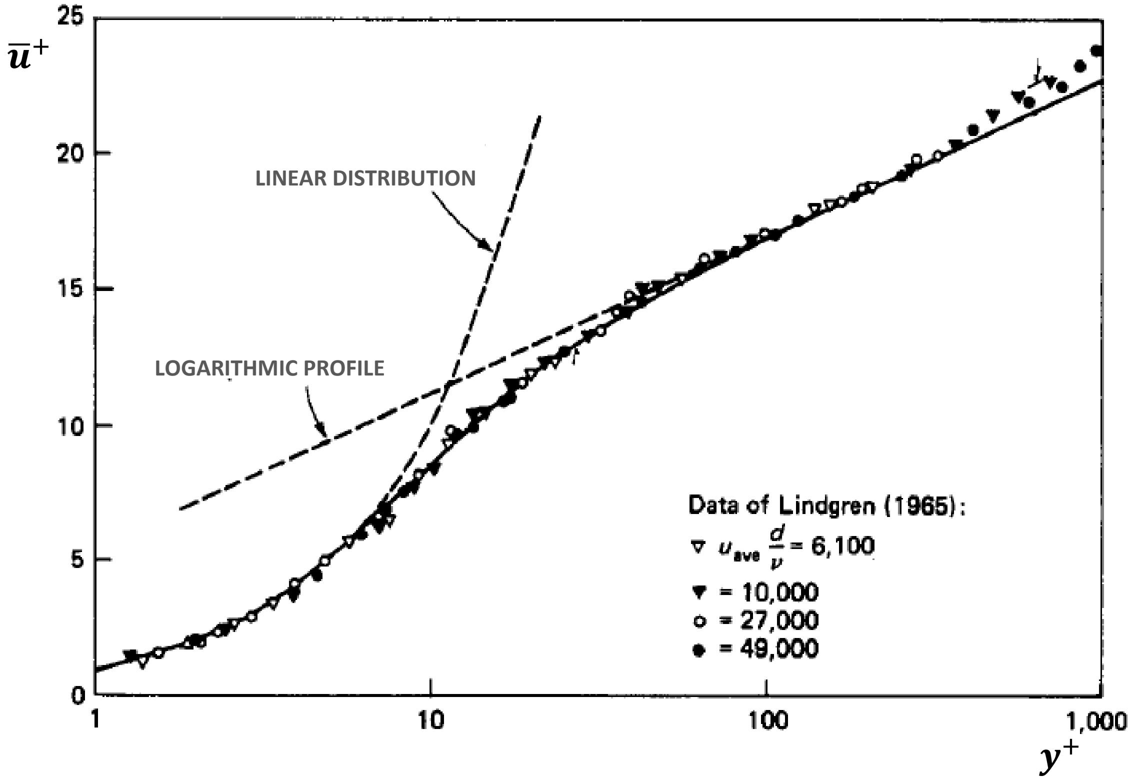


Fig. 3.2.- Experimental data and velocity distributions for the viscous sublayer (Eq. 3.11) and turbulent region (Eq. 3.7).

enough to preclude the existence of a region. As the roughness is much greater than the viscous sublayer, the resistance comes mainly from the drag of the protruding elements on the surface. The idea of both kind of walls is sketched in Fig. 3.3. There is a transition from the hydraulically smooth to the rough surface, in which both viscous stress and drag are important in the generation of flow resistance. This originates a wall in the transition smooth-rough regime.

A question regarding how the roughness size is determined. It is not measured directly. Actually it is an *equivalent roughness*. The size ε is the result of a distribution of sizes that gives the same resistance to the flow than that obtained by Nikuradse in experiments with uniform sand grains of size k_S glued to the inner wall of pipes, as shown in Fig. 3.4. Although both kind of roughness have the same effects for the hydraulically smooth and rough walls, they present a difference in the transition. For Nikuradse's uniform grains, there is a sudden passage from smooth to rough wall. In the case of ε, there is a gentle transition from smooth to rough due to the several sizes involved in the distribution roughness heights. The value ε corresponds to that obtained for commercial pipes.

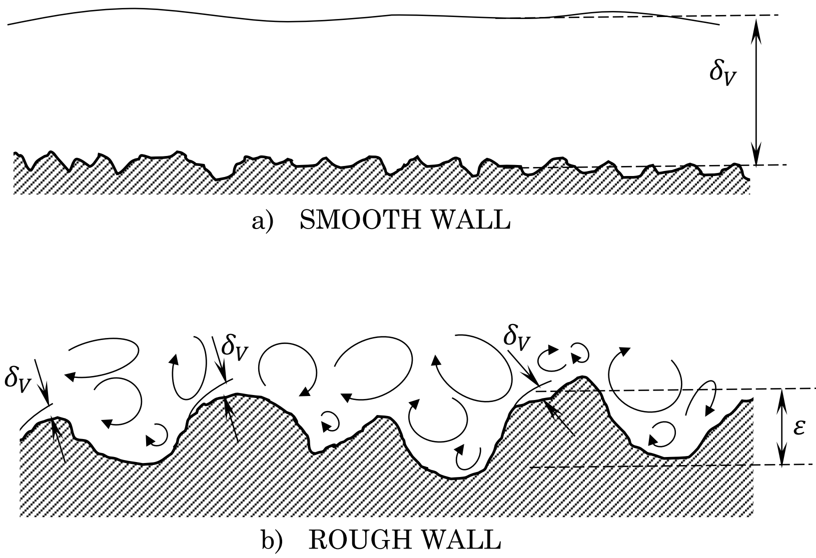


Fig. 3.3.- Definition of hydraulically smooth and rough surfaces.

a) On a smooth wall, the viscous sublayer cover completely the roughness of the wall. The flow in the turbulent region is not influenced by the size of the roughness and the flow resistance is due to the viscous shear acting in the viscous sublayer.

b) The viscous sublayer is completely destroyed by the roughness elements and the flow resistance is originated by the drag exerted by them.

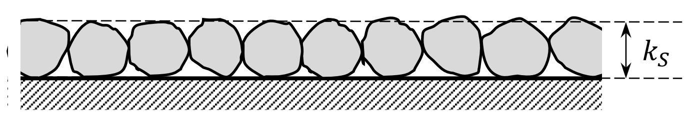


Fig. 3.4.- Nikuradse's roughness. Sand grains were glued to the wall of the

Based on the limits of the different regions indicated before, the hydrodynamic type of wall is defined depending on which of them the roughness is contained. It is expressed in terms of the roughness size made dimensionless with the inner variables:

$$k_S^+ = \frac{k_S u_*}{\nu} \tag{3.13}$$

Due to the arbitrariness in the definition of the region boundaries, the values assigned to k_S^+ to define the type of wall vary slightly among different authors, as shown in Table 3.1,although those assigned by Prandtl are the most used in practice.

Table 3.1.- DIMENSIONLESS ROUGHNESS SIZE AND HYDRAULIC TYPE OF WALL

HYDRAULIC TYPE OF WALL	PRANDTL (Schlichting, 1979)	WHITE (White, 1991)
SMOOTH	$k_S^+ < 5$	$k_S^+ < 4$
TRANSITION SMOOTH -ROUGH	$5 \leq k_S^+ \leq 70$	$4 \leq k_S^+ \leq 60$
ROUGH	$k_S^+ > 70$	$k_S^+ > 60$

VELOCITY DISTRIBUTION IN THE FULLY TURBULENT REGION ($\boldsymbol{y^+ > 70}$)

As it was indicated, the velocity distribution in the fully turbulent region is directly related to the flow resistance mechanism, which depends on the type of wall. Thus, the distance y_0 at which the boundary condition is evaluated should consider it. Eq. 3.1 can be written as:

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ln\left(\frac{y}{y_0}\right) \tag{3.14}$$

The type of wall should be taken into account in the value of y_0.

Hydraulically smooth wall ($\boldsymbol{k_S^+ < 5}$)

In this case, we already found that $y_0 \sim \nu/u_*$ (Eq. 3.4) and Eq. 3.7 is valid, which usually is written as:

$$\bar{u}^+ = \frac{1}{\kappa}\ln(y^+) + B \tag{3.15}$$

κ and B are determined from experimental data, such as those presented in Fig. 3.2, resulting $\kappa = 0.4$ and $B = 5.5$ (Nikuradse, 1933).

Hydraulically rough wall ($\boldsymbol{k_S^+ > 70}$)

The flow resistance mechanism is due to the drag generated by the grains large enough to preclude the existence of a viscous sublayer. In this case, it is relevant the size of the roughness, so $y_0 \sim k_S$ and Eq. 3.14 becomes:

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ln\left(\frac{y}{k_S}\right) + C \tag{3.16}$$

C is determined from experimental measurements like that presented in Fig. 3.5, which correspond to the data taken by Nikuradse (1933), and takes the value 8.5.

Wall in smooth-rough transition **($5 \leq k_S^+ \leq 70$)**

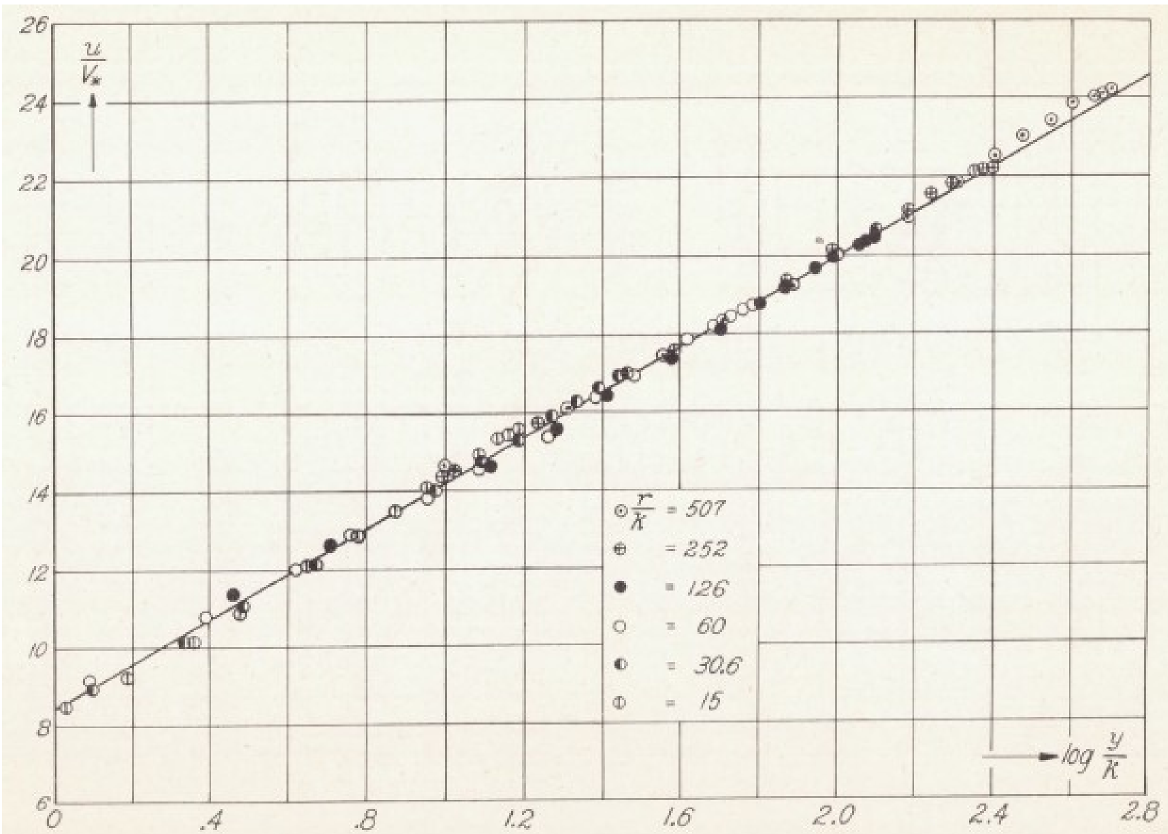


Fig. 3.5.- Velocity distribution in a hydraulically rough pipe. Note that in the horizontal axis is $\log_{10}\left(\frac{y}{k_S}\right)$ (Nikuradse, 1933).

I this case, the flow resistance is the result of both viscous and drag effects. Thus, a functional relationship $y_0 = f(\mu, \rho, \tau_0, k_S)$ should exist. From the Buckingam Π theorem we obtain:

$$\Pi_1 = \frac{y_0}{k_S} \quad , \quad \Pi_2 = \frac{u_* k_S}{\nu} \tag{3.17}$$

The dimensionless relation between both dimensionless parameters is written as:

$$\Phi\left(\frac{y_0}{k_S}, k_S^+\right) = 0 \tag{3.18}$$

Which is the same that:

$$\frac{y_0}{k_S} = \Phi_1(k_S^+) \tag{3.19}$$

Thus, y_0 is given by:

$$y_0 = k_S \Phi_1(k_S^+) \tag{3.20}$$

Replacing y_0 from Eq. 3.20 into Eq. 3.14:

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ln\left(\frac{y}{k_S}\right) + A(k_S^+) \tag{3.21}$$

The function $A(k_S^+)$ is determined computing the deviation shown by the experimental data corresponding to the smooth-rough transition with respect to the velocity distribution for the rough wall (Eq. 3.16), resulting the graphic relationship shown in Fig. 3.6 (Nikuradse, 1933).

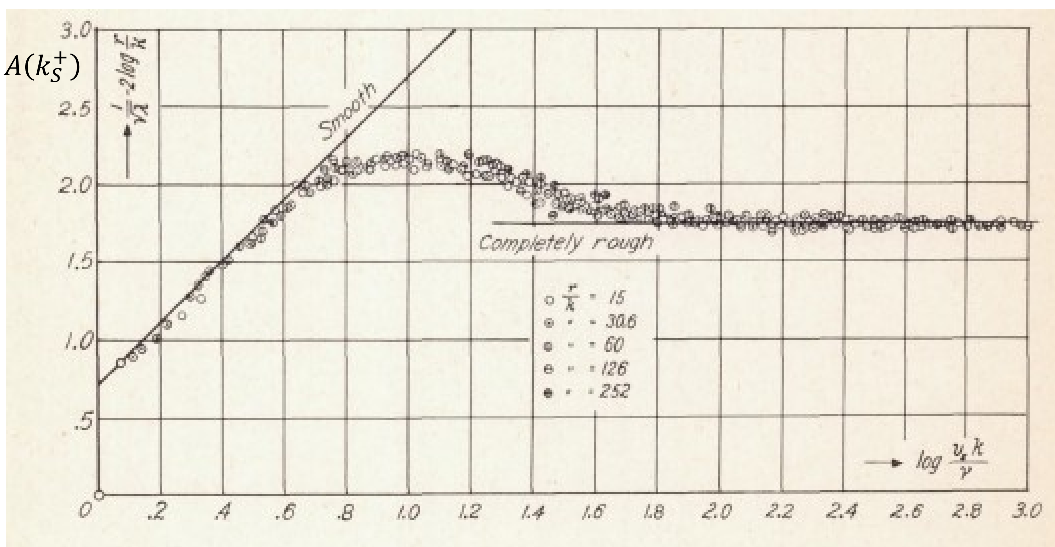


Fig. 3.6.- Function $A(k_S^+)$ of the velocity distribution in fully turbulent region of a pipe with wall in the transition smooth-rough regime. Note that in the horizontal axis is $\log_{10}(k_S^+)$ (Nikuradse, 1933).

HYDRAULICALLY SMOOTH WALL ($k_S^+ < 5$). Velocity distribution in the buffer layer ($5 \leq y^+ \leq 70$)

In hydraulically smooth walls, a buffer layer matching the viscous sublayer with the fully turbulent region exists. Spalding (1961) gives two expressions for the velocity profile in the buffer layer. The difference between them is the presence of a term raised to the fourth power. However, the equation without this term adjust very well to the experimental data, as shown by White (1991, Fig. 6.11). The equation is:

$$y^+ = u^+ + e^{-\kappa B}\left[e^{\kappa u^+} - 1 - \kappa u^+ - \frac{(\kappa u^+)^2}{2} - \frac{(\kappa u^+)^3}{6}\right] \tag{3.22}$$

Note that Eq. 3.22 is not explicit in u^+, with $\kappa = 0.4$ and $B = 5.5$. The range of validity of this equation extends further than the buffer layer, spanning $0 \leq y^+ < 300$.

The different velocity profiles arising in the turbulent flow are summarized in Table 3.2.

Table 3.2.- SUMMARY OF VELOCITY DISTRIBUTIONS IN TURBULENT FLOWS

TYPE OF WALL	VELOCITY DISTRIBUTION
HYDRAULICALLY SMOOTH $k_S^+ < 5$	Viscous sublayer, $y^+ < 5$ $u^+ = y^+$
	Buffer layer, $5 \leq y^+ \leq 70$ Eq. 3.22
	Fully turbulent region, $y^+ > 70$ $u^+ = \frac{1}{\kappa}\ln(y^+) + 5.5$
TRANSITION SMOOTH-ROUGH $5 \leq k_S^+ \leq 70$	$u^+ = \frac{1}{\kappa}\ln\left(\frac{y}{k_S}\right) + A(k_S^+)$ $A(k_S^+)$ from Fig. 3.6
HYDRAULICALLY ROUGH $k_S^+ > 70$	$u^+ = \frac{1}{\kappa}\ln\left(\frac{y}{k_S}\right) + 8.5$

FRICTION COEFFICIENT OF PIPES WITH TURBULENT FLOWS

The friction coefficient f, necessary to compute the frictional energy loss (Eq. 47), can easily be computed from the velocity distributions presented in the previous sections. Eq. 68 gives the relation between f and the cross-sectional mean velocity, U:

$$\frac{u_*}{U} = \sqrt{\frac{f}{8}} \tag{68}$$

The velocity distributions have the form $\bar{u}/u_*$ as a function of the distance from the wall. It looks evident that, integrating the velocity distribution across the flow section will allows to obtain U/u_*, hence the friction factor f. The analysis that follows will consider cylindrical pipes because in this case exists an extensive set of data that permits to confirm the validity of the expressions obtained. The results can be generalized to other geometries.

Friction factor for turbulent flow with hydraulically smooth walls

The velocity distribution in the fully turbulent region is given by Eq. 3.15:

$$\bar{u}^+ = \frac{1}{\kappa}\ln(y^+) + B \tag{3.15}$$

The cross-sectional mean velocity is computed from

$$U = \frac{1}{A}\int_A \bar{u}\ dA \tag{3.23}$$

For a pipe with inner diameter D, $A = \pi D^2/4$. The distance from the wall (y) is related to the radial distance from the axis of the pipe (r) trough $y + r = D/2$. The element of surface is $dA = 2\pi r dr$. In order to simplify the computations, we will neglect the viscous sublayer and buffer regions, i.e., we will assume that Eq. 3.15 is valid in all the flow domain. Thus, from Eq.s 3.15 and 3.23:

$$U = \frac{4}{\pi D^2}\int_o^{\frac{D}{2}} \left(\frac{u_*}{\kappa}\ln\left(\left(\frac{D}{2} - r\right)\frac{u_*}{\nu}\right) + u_* B\right) 2\pi r dr \tag{3.24}$$

$$U = \frac{8}{D^2}\frac{u_*}{\kappa}\int_o^{\frac{D}{2}} \left(\ln\left(\frac{D}{2} - r\right) + \ln\left(\frac{u_*}{\nu}\right) + \kappa B\right) r dr \tag{3.25}$$

$$U = \frac{8}{D^2}\frac{u_*}{\kappa}\left[\int_o^{\frac{D}{2}} \ln\left(\frac{D}{2} - r\right) r dr + \left(\ln\left(\frac{u_*}{\nu}\right) + \kappa B\right)\int_o^{\frac{D}{2}} r dr\right] \tag{3.26}$$

$$U = \frac{8}{D^2}\frac{u_*}{\kappa}\left[\int_o^{\frac{D}{2}} \ln\left(\frac{D}{2} - r\right) r dr + \left(\ln\left(\frac{u_*}{\nu}\right) + \kappa B\right)\frac{1}{2}\left(\frac{D}{2}\right)^2\right] \tag{3.27}$$

A comment regarding the integral in Eq. 3.27. It will give a term containing $(D - 2r)\ln(D - 2r)$that when evaluated at $r = D/2$ gives $0 \times (-\infty)$. To overcome this problem, l'Hôpital rule must be applied. Grouping the sum of logarithms in the logarithm of the product, the following result is obtained:

$$U = \frac{u_*}{\kappa}\left[\left(\ln\left(\frac{1}{2}\frac{Du_*}{\nu}\right) - \frac{3}{2}\right) + \kappa B\right] \tag{3.28}$$

We can rearrange Eq. 3.28 to give the structure of Eq. 68:

$$\frac{u_*}{U} = \sqrt{\frac{f}{8}} = \frac{1}{\frac{1}{\kappa}\left(\ln\left(\frac{1}{2}\frac{Du_*}{\nu}\right) - \frac{3}{2}\right) + B} \tag{3.29}$$

Generally, the flow mean velocity $U = Q/A$ is known, and not the shear velocity, and it is convenient to change the argument of the logarithm in Eq. 3.29 multiplying and diving it by U:

$$\frac{Du_*}{\nu} = \frac{DU}{\nu}\frac{u_*}{U} = Re\sqrt{\frac{f}{8}} \tag{3.30}$$

Replacing Eq. 3.30 in Eq. 3.29 and working with its reciprocal:

$$\frac{1}{\sqrt{f}} = \frac{1}{\sqrt{8}\kappa}\left[\ln\left(Re\sqrt{f}\right) - \ln\left(\sqrt{32}\right) - \frac{3}{2} + \kappa B\right] \tag{3.31}$$

Using the values $\kappa = 0.4$ and $B = 5.5$:

$$\frac{1}{\sqrt{f}} = 0.884\ln\left(Re\sqrt{f}\right) - 0.913 \tag{3.32}$$

Eq. 3.32 is usually expressed in terms of the decimal logarithm:

$$\frac{1}{\sqrt{f}} = 2.035\log\left(Re\sqrt{f}\right) - 0.913 \tag{3.33}$$

Eq. 3.33 was obtained by Prandtl in 1935 (White, 1991). As the buffer region and viscous sublayer were neglected in its deduction, the numerical constants that appear in it were re-computed in order to fit the experimental data taken by Nikuradse (1933), resulting:

$$\frac{1}{\sqrt{f}} = 2\log\left(Re\sqrt{f}\right) - 0.8 \tag{3.34}$$

This is Prandtl's universal law of friction for smooth pipes (Schlichting, 1979) and it is not have limitations regarding the Reynolds number (within the turbulent regime). Computation of f from Eq. 3.34 must be done graphically or numerically, usually by iteration. A simpler equation is the empirical relationship presented by Blasius in 1913, but limited to $3 \times 10^3 \leq Re \leq 10^5$ (Blasius, 1913, p. 12)

$$f = \frac{0.3164}{Re^{1/4}} \tag{3.35}$$

Friction factor for turbulent flow with hydraulically rough walls

The friction factor for turbulent flows with rough walls is obtained from the velocity distribution given by Eq. 3.16.

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ln\left(\frac{y}{k_S}\right) + C \tag{3.16}$$

Assuming eq. 3.16 valid in all the cross-section, computing the average velocity U , rearranging the final result to form $\sqrt{f} = \sqrt{8}\,u_*/U$, and using $\kappa = 0.4$ and $C = 8.5$, the following expression is obtained:

$$\frac{1}{\sqrt{f}} = 0.942\ln\left(\frac{D}{2k_S}\right) + 1.68 \tag{3.36}$$

In terms of the decimal logarithm:

$$\frac{1}{\sqrt{f}} = 2.2\log\left(\frac{D}{2k_S}\right) + 1.68 \tag{3.36}$$

The numerical constants are adjusted to the experimental data, resulting:

$$\frac{1}{\sqrt{f}} = 2\log\left(\frac{D}{2k_S}\right) + 1.74 \tag{3.37}$$

Frequently, the additive constant 1.74 is introduced into the logarithm argument, giving:

$$\frac{1}{\sqrt{f}} = 2\log\left(3.7\frac{D}{k_S}\right) \tag{3.38}$$

Friction factor for turbulent flow with walls in transition smooth-rough

An analytical derivation of a relationship for the friction factor for the turbulent flow with walls in the transition smooth-rough regime is not possible because we do not have a simple analytical equation for $\bar{u}/u_*$. In this case, the velocity profile is given by Eq. 3.21, with the additive constant depending of k_S^+, given in Fig. 3.6. In order to determine a relation for f, we will analyse the deviation of the experimental data from the friction factor associated to a hydraulically rough wall. In order to avoid confusion with the friction factors, we will use the sub-indexes S to denote the smooth wall and R for the rough. We can write for smooth wall friction coefficient (Eq. 3.34):

$$2\log\left(Re\sqrt{f_S}\right) - \frac{1}{\sqrt{f_S}} = 0.8 \tag{3.39}$$

$$2\log\left(\frac{UD}{\nu}\sqrt{8}\frac{u_*}{U}\right) - \frac{1}{\sqrt{f_S}} = 0.8 \tag{3.40}$$

$$2\log\left(\frac{u_*D}{\nu}\sqrt{8}\right) - \frac{1}{\sqrt{f_S}} = 0.8 \tag{3.41}$$

$$2\log\left(\frac{u_*k_S}{\nu}\frac{D}{k_S}\sqrt{8}\right) - \frac{1}{\sqrt{f_S}} = 0.8 \tag{3.42}$$

$$2\log\left(\frac{D}{k_S}\right) + 2\log(k_S^+) + 2\log(\sqrt{8}) - \frac{1}{\sqrt{f_S}} = 0.8 \tag{3.43}$$

$$2\log\left(3.7\frac{D}{k_S}\right) - 2\log(3.7) + 2\log(k_S^+) + 2\log(\sqrt{8}) - \frac{1}{\sqrt{f_S}} = 0.8 \tag{3.44}$$

We identify the first term of Eq. 3.44 as $1/\sqrt{f_R}$ (Eq. 3.38):

$$\frac{1}{\sqrt{f_R}} - 2\log(3.7) + 2\log(k_S^+) + 2\log(\sqrt{8}) - \frac{1}{\sqrt{f_S}} = 0.8 \tag{3.45}$$

$$\frac{1}{\sqrt{f_R}} - \frac{1}{\sqrt{f_S}} = 0.8 + 2\log(3.7) - 2\log(\sqrt{8}) - 2\log(k_S^+) \tag{3.46}$$

$$\frac{1}{\sqrt{f_R}} - \frac{1}{\sqrt{f_S}} = 1.033 - 2\log(k_S^+) \tag{3.47}$$

Eq. 3.47 is the deviation (in terms of $1/\sqrt{f}$) of the friction factor for smooth walls from the friction factor for rough walls. Now, we will define the deviation of the friction factor for any type of wall from that associated to the rough surface as:

$$M = 2\log\left(3.7\frac{D}{k_S}\right) - \frac{1}{\sqrt{f}} \tag{3.48}$$

As Eq. 3.48 is valid for any type of wall, M should be function of k_S^+. Using the experimental data presented by Nikuradse (1933) is possible to know $M(k_S^+)$. In effect, with the data is possible to generate the Table. 3.3. The first four columns contains Nikuradse's data and the fifth is computed using Eq. 3.48.

Table 3.3.- ORGANIZATION OF NIKURADSE'S DATA TO GENERATE THE FUNCTION $M(k_S^+)$

Re	$\frac{D}{k_S}$	f	$k_S^+ = \frac{k_S u_*}{\nu}$	M
:	:	:	:	:
:	:	:	:	:
:	:	:	:	:

The relation between M and k_S^+ is presented in Fig. 3.7, taken from the article by Colebrook and White (1937) and corresponds to line labelled "Nikuradse sanded". The straight line labelled "smooth law" is Eq. 3.47. Obviously, for rough walls $M = 0$ ("rough law" in the figure).

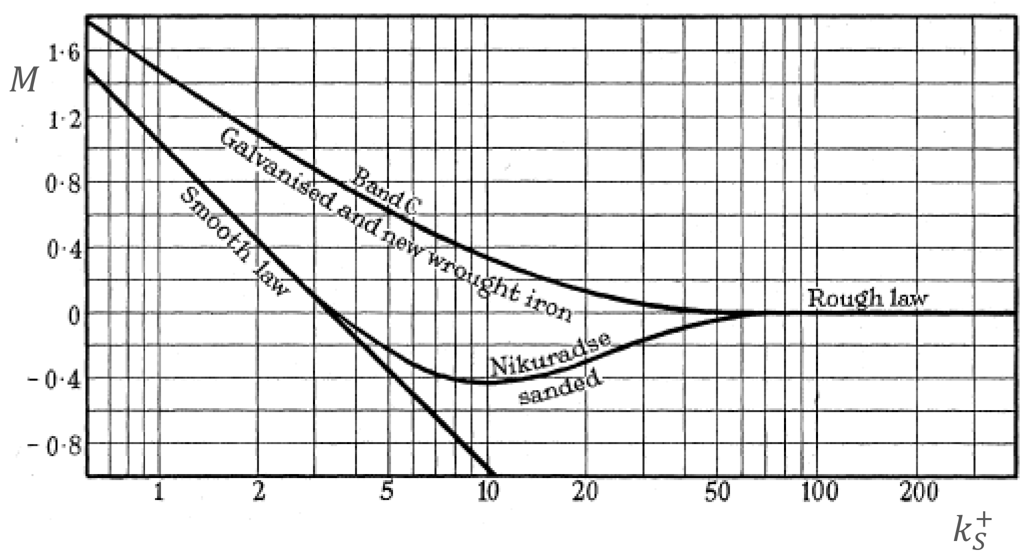


Fig. 3.7.- Deviation of $1/\sqrt{f}$ from the rough wall. The function M covers all type of walls.

The objective of the article by Colebrook and White (1937) was "to determine how the nature of the roughness influenced the transition". To accomplish that goal, sand of different sizes and following some particular arrangements were glued to the pipe. But they also added information of commercial pipes reported by other authors (Freeman, quoted by Mills, 1923; and Heywood, 1924). The commercial pipes generate the curve labelled as "Galvanised and new wrought iron" in Fig. 3.7. The different behaviour among the measurements carried out in commercial pipes and those obtained by Nikuradse is result of the non-uniformity of the roughness size in the commercial pipes. In this case, protuberances larger than the average disturb and destroy the viscous sublayer before that the rest of the protuberances. This is a gradual process that makes an earlier and gradual separation of the M curve from the smooth case when compared with pipes of uniform roughness.

As it was indicated before, it is customary to denote by ε the equivalent roughness that arise in pipes with non-uniform size of protuberances. A dimensionless equivalent roughness is defines as:

$$\varepsilon^+ \equiv \frac{\varepsilon u_*}{\nu} \tag{3.49}$$

Thus, for commercial pipes, the experimental relationship between M and ε^+ is found to be:

$$M = 2\log\left(1 + \frac{3.29}{\varepsilon^+}\right) \tag{3.50}$$

It is easy to verify that Eq. 3.50 does not have limitations regarding the type of wall, covering all of them. For smooth walls, ε^+ is very small, and $3.29/\varepsilon^+ \gg 1$, and $M \approx 2\log(3.29/\varepsilon^+) = 2\log(3.29) - 2\log(\varepsilon^+)$, that is to say:

$$M = 1.033 - 2\log(\varepsilon^+) \tag{3.51}$$

Eq. 3.51 is in agreement with Eq. 3.47, the deviation for smooth surfaces. In the same way, for hydraulically rough walls, ε^+ is large, resulting $3.29/\varepsilon^+ \ll 1$, and $M \approx 2\log(1) = 0$, i.e., there is not deviation from the friction factor for rough surface.

Thus, it is possible to determine an expression for the friction factor for turbulent flows, valid for all type of walls. In effect, replacing Eq. 3.50 into eq. 3.48:

$$2\log\left(1 + \frac{3.29}{\varepsilon^+}\right) = 2\log\left(3.7\frac{D}{\varepsilon}\right) - \frac{1}{\sqrt{f}} \tag{3.52}$$

But

$$\varepsilon^+ = \frac{\varepsilon u_*}{\nu} = \frac{\varepsilon}{D}\frac{D u_*}{\nu} = \frac{\varepsilon}{D}\frac{DU}{\nu}\frac{u_*}{U} = \frac{\varepsilon}{D}Re\sqrt{\frac{f}{8}} \tag{3.53}$$

Replacing Eq. 3.53 in Eq. 3.52 results:

$$-\frac{1}{\sqrt{f}} = 2\log\left(1 + \frac{3.29\sqrt{8}}{Re\sqrt{f}}\frac{D}{\varepsilon}\right) - 2\log\left(3.7\frac{D}{\varepsilon}\right) \tag{3.54}$$

$$-\frac{1}{\sqrt{f}} = 2\log\left[\left(1 + \frac{3.29\sqrt{8}}{Re\sqrt{f}}\frac{D}{\varepsilon}\right)\frac{\varepsilon}{3.7D}\right] \tag{3.55}$$

$$-\frac{1}{\sqrt{f}} = 2\log\left(\frac{\varepsilon}{3.7D} + \frac{3.29\sqrt{8}}{3.7}\frac{1}{Re\sqrt{f}}\right) \tag{3.56}$$

Finally, the friction factor for turbulent flows is pipes is given by:

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{\varepsilon}{3.7D} + \frac{2.51}{Re\sqrt{f}}\right) \tag{3.57}$$

Eq. 3.57 was presented in an article written by Colebrook in 1939. Although he is the only author of the article, Colebrook recognizes White's contribution in the development of the formula, which is known as the Colebrook-White equation. Eq. 3.57 is not explicit in f, therefore computation must be done iteratively, which is not a problem with the calculators existing nowadays. However, in the 1940s, it resulted a cumbersome task when it had to be computed manually o with slide rules. Thus, the graphic relationship shown in Fig. 3.8 was presented by Rouse (1943). The graph contains a set of curves with the relative roughness (ε/D) as parameter. Depending on the known variables, the diagram can be accessed from the lower horizontal axis, which contains $Re\sqrt{f}$. Considering the Darcy-Weisbach equation (Eq.47), $f = 2JgD/U^2$, the term $Re\sqrt{f}$ can be written as: $Re\sqrt{f} = UD/\nu\sqrt{2JgD/U^2} = \sqrt{2Jg}\ D^{3/2}/\nu$, which is independent of the velocity. Thus, if the viscosity, gradient of the energy loss (J), pipe diameter and roughness are known the velocity (and hence the discharge) are determined directly. On the other hand, if the velocity and pipe characteristics are known and the friction factor has to be computed, the Reynolds number can be determined and the graph is accessed from the upper horizontal axis until reach the line corresponding to the relative roughness and f (or $1/\sqrt{f}$) is read in the vertical axis. Note that Re follow curved lines. (In Rouse's notation, $S = J$).

However, the most known graphic relationship is that due to Moody (1944), who only differs from Rouse's in the principal axis. Moody used the Reynolds number in the lower horizontal axis. Thus, we have to follow a vertical straight line for a given Re (and not curved lines as in Rouse diagram). Note that Moody indicates that the units of velocity, diameter and kinematic viscosity must be ft/s, ft and ft^2/s, respectively. Obviously, this is not necessary. As Re is dimensionless, the only requirement is that the units must be in any coherent system of units.

Note that we have presented relationships for the friction factor for two flow regimes: laminar ($f = 64/Re$) and turbulent (Eq. 3.57). We have not presented a relation for the transition laminar-turbulent regime because it is very uncertain and small variations of Re mean large variations in f. In general, flows in the transition regime are not usual in civil or environmental engineering.

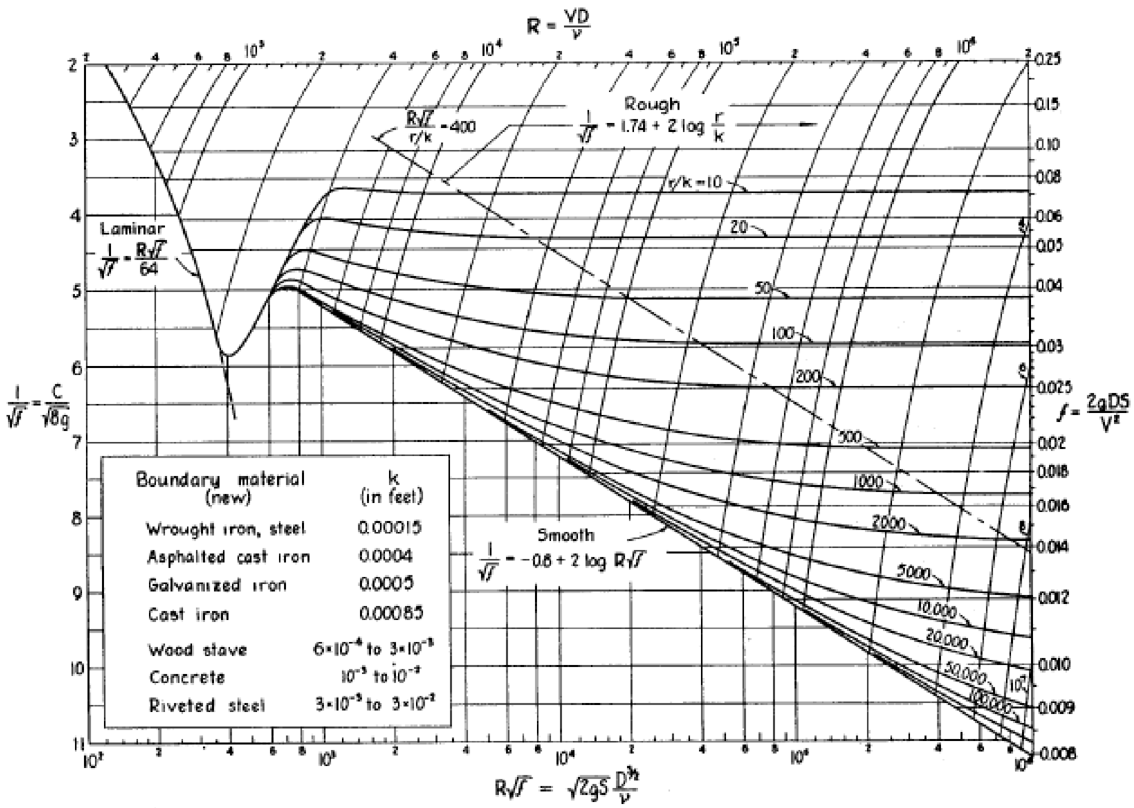


Fig. 3.8.- Rouse diagram (Rouse, 1943)

A large amount of equations for f in the turbulent regime can be found in the literature. Many of them are experimental relationships determined for specific pipes and materials. Also, there a many relationships that are fittings to the Moody diagram (Eq.3.57), with only goal to have explicit expressions for f. Some of them are the following (Beluco and Camano, 2016):

Haaland:

$$\frac{1}{\sqrt{f}} = -1.8 \log\left[\left(\frac{\varepsilon}{3.7D}\right)^{1.1} + \frac{6.9}{Re}\right] \tag{3.58}$$

Barr:

$$\frac{1}{\sqrt{f}} = -2 \log\left[\frac{\varepsilon}{3.7D} + \frac{5.15}{Re^{0.892}}\right] \tag{3.59}$$

Eck:

$$\frac{1}{\sqrt{f}} = -2\log\left[\frac{\varepsilon}{3.715D} + \frac{15}{Re}\right] \tag{3.60}$$

As the above equations (an others similar) are approximations to Eq. 3.57, the friction factor computed with them will have an error.

Finally in Table 3.4, typical values of the roughness size for some materials are presented.

Table 3.4.- TYPICAL VALUES OF THE ROUGHNESS SIZE

MATERIAL	ε (mm)
Riveted steel	0.92 - 9.2
Concrete	0.31 - 3.1
Ductile iron	2.6
Wood stave	0.09 - 0.18
Galvanized iron	0,15
Cast iron – asphalt dipped	0,12
Cast iron uncoated	0,25
Carbon steel or wrought iron	0,045
Stainless steel	0,045
Fiberglass	0,005
Drawn tubing – glass, brass, plastic	0,0015
Copper	0,0015
Aluminium	0,0015
PVC	0,0015
Red brass	0,0015

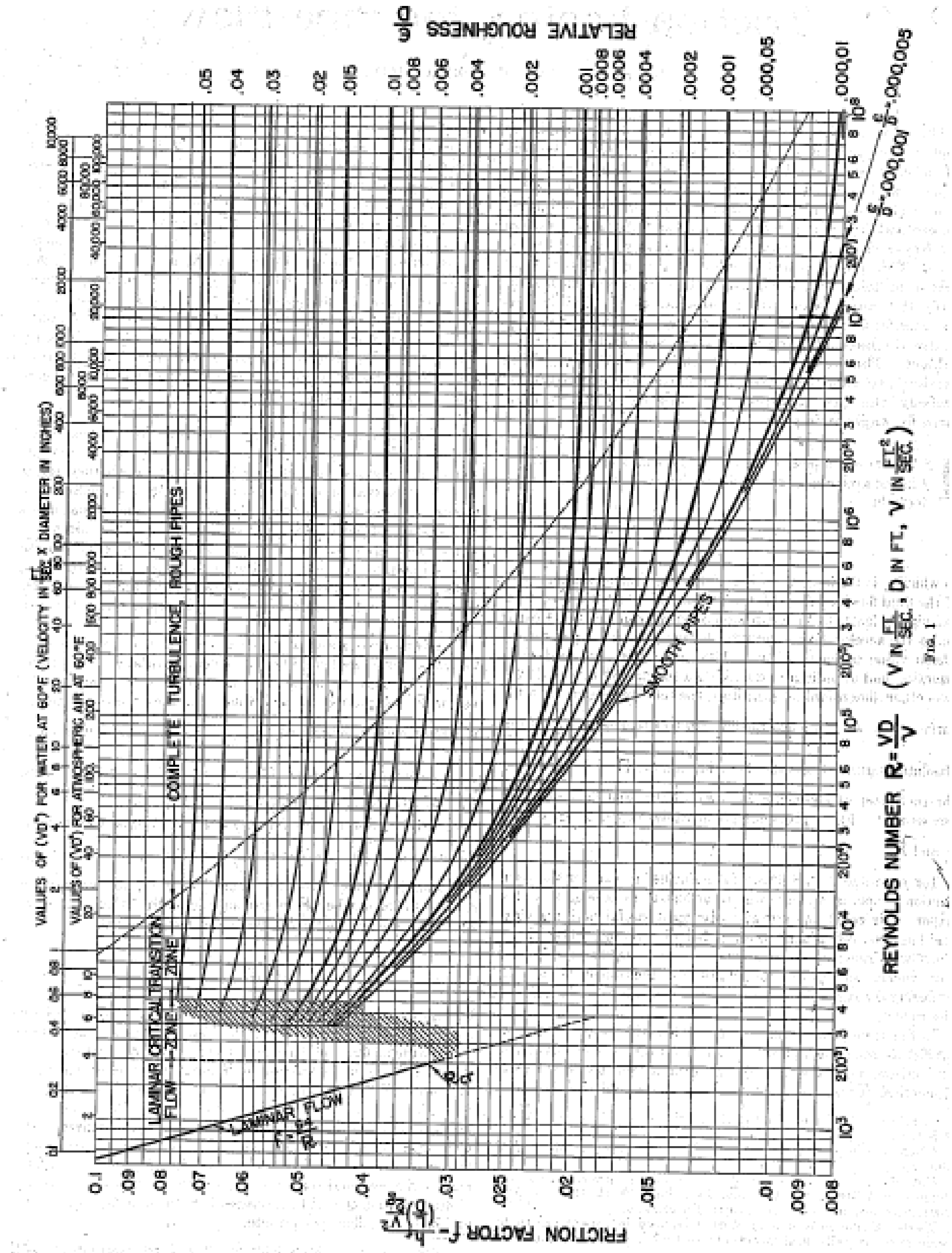


Fig. 3.9.- Moody's digram (Moody, 1944)

RESISTANCE LAWS IN OPEN CHANNEL FLOWS

The application of relationships like Eq. 3.57 has been extended to the flow in open channels, adjusting the numerical coefficients with *ad hoc* experimental data (Yen, 2002), and replacing the pipe diameter D by the hydraulics radius $R_H = A/\chi$, where A is the flow area and χ is the wetted perimeter. Eq. 3.57 is transformed into:

$$\frac{1}{\sqrt{f}} = -K_1 \log\left(\frac{k_S}{K_2 R} + \frac{K_3}{4Re_{RH}\sqrt{f}}\right) \tag{3.61}$$

In Eq. 3.61, Re_{RH} is the Reynolds number based on the hydraulic radius, i.e., $Re_{RH} = UR_H/\nu$. K_1, K_2 and K_3 are coefficients fitted by different authors. Yen (2002) presents the summary shown in Table 3.5.

Table 3.5.- COEFFICIENTS OF EQ. 3.57 ACCORDING TO DIFFERENT AUTHORS (YEN, 2002)

CHANNEL GEOMETRY	REFERENCE	K_1	K_2	K_3	REMARKS
Full circular pipe	Colebrook (1939)	2.0	14.83	2.52	
Wide channel	Keulegan (1938)	2.03	11.09	3.41	
Wide channel	Rouse (1946, p. 214)	2.03	10.95	1.70	
Wide channel	Thijsse (1949)	2.03	12.2	3.033	
Wide channel	Sayre and Albertson (1961)	2.14	8.888	7.17	
Wide channel	Henderson (1966)	2.0	12.0	2.5	
Wide channel	Graf (1971, p. 305)	2.0	12.9	2.77	
Wide channel	Reinius (1961)	2.0	12.4	3.4	
Rectangular	Reinius (1961)	2.0	14.4	2.9	Width/depth=4
Rectangular	Reinius (1961)	2.0	14.8	2.8	Width/depth=2
Rectangular	Zegzhda (1938)	2.0	11.55	0	Dense sand

When Eq. 3.61 is applied to natural channels (with fixed boundaries), it is also necessary to define the roughness, k_S. The bed of natural channels, when it is formed by granular non-cohesive sediments, usually contains a wide range of particle sizes. The matter of what is the appropriate size to represent k_S depends on the researcher, and usually is expressed in terms of the diameter d_X of the size distribution (d_X is the size of the particle under which it is found the X% of the sediment):

$$k_S = \alpha_S d_X \tag{3.62}$$

Yen (2002) gives a list of the diameter d_X used by different authors and the corresponding value of the coefficient α_S, which is reproduced in Table 3.6.

Table 3.6.- EQUIVALENT SIZE ROUGHNESS ACCORDING TO DIFFERENT AUTHORS (YEN, 2002)

RESEARCHER	CHARACTERISTIC SIZE	α_s
Ackers and White (1973)	d_{35}	1.23
Strickler (1923)	d_{50}	3.3
Keulegan (1938)	d_{50}	1
Meyer-Peter and Muller (1948)	d_{50}	1
Thompson and Campbell (1979)	d_{50}	2.0
Hammond et al. (1984)	d_{50}	6.6
Einstein and Barbarossa (1952)	d_{65}	1
Irmay (1949)	d_{65}	1.5
Engelund and Hansen (1967)	d_{65}	2.0
Lane and Carlson (1953)	d_{75}	3.2
Gladki (1979)	d_{80}	2.5
Leopold et al. (1964)	d_{84}	3.9
Limerinos (1970)	d_{84}	2.8
Mahmood (1971)	d_{84}	5.1
Hey (1979), Bray (1979)	d_{84}	3.5
Ikeda (1983)	d_{84}	1.5

Colosimo et al. (1986)	d_{84}	3 - 6
Whiting and Dietrich (1990)	d_{84}	2.95
Simons and Richardson (1966)	d_{85}	1
Kamphuis (1974)	d_{90}	2.0
van Rijn (1982)	d_{90}	3.0

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