Pauta Auxiliar #10

P1) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) \doteq x^2y + x\cos(y)$

Para calcular su serie de Taylor, basta con que recordemos la fórmula:

$$f(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0)^T \begin{pmatrix} x-x_0 \\ y-x_0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}^T \mathcal{H}_f(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + R_2(x,y),$$

con $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{|R_2(x,y)|}{\left\|\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}\right\|^2} = 0$. Hacemos $(x_0,y_0) = (1,0)$

- $f(\cancel{x_0},\cancel{y_0}) = 1$
- $\nabla f(\cancel{x_0},\cancel{y_0}) = \begin{pmatrix} 2xy + \cos(y) \\ x^2 - x\,\text{sen}(y) \end{pmatrix}\Bigg|_{(1,0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- $\mathcal{H}_f(1,0) = \begin{pmatrix} 2y & 2x - \text{sen}(y) \\ 2x - \text{sen}(y) & -x\cos(y) \end{pmatrix}\Bigg|_{(1,0)} = \begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}$

$\Rightarrow \cancel{f(x,y) \approx 1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix}^t \begin{pmatrix} x \\ y \end{pmatrix} +}$

$$\Rightarrow f(x,y) \approx 1 + (1,1)\begin{pmatrix} x-1 \\ y \end{pmatrix} + \frac{1}{2}(x-1,y)\begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x-1 \\ y \end{pmatrix}$$

$$= 1 + x - 1 + y + \frac{1}{2}(x-1,y)\begin{pmatrix} 2y \\ 2x-2-y \end{pmatrix}$$

$$= \cancel{1} + x \cancel{-1} + y + \frac{1}{2}\left[2xy - 2y + 2xy - 2y - y^2\right]$$

$$= x - y + 2xy - \frac{1}{2}y^2$$

(*) Como f es claramente C^2, tenemos derecho a escribir su serie de Taylor.

P2|

$$f(x,y) \doteq \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

a) Calcule $\partial_x f(x,y)$ y $\partial_y f(x,y)$, para $(x,y) \neq (0,0)$

Dm:

- $$\frac{\partial f}{\partial x}(x,y) = \frac{[y(x^2-y^2) + xy \cdot 2x](x^2+y^2) - 2x^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$

- $$\frac{\partial f}{\partial y}(x,y) = \frac{[x(x^2-y^2) - xy \cdot 2y](x^2+y^2) - 2y^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$$

b) Muestre que $\partial_x f(0,0) = \partial_y f(0,0) = 0$

Dm:

- $$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{f(t,0) - \cancelto{0}{f(0,0)}}{t} = \lim_{t \to 0} \frac{1}{t} \cdot \frac{t \cdot 0 \cdot (t^2-0^2)}{t^2+0^2} = 0$$

- $$\frac{\partial f}{\partial y}(0,0) = \lim_{t \to 0} \frac{f(0,t) - \cancelto{0}{f(0,0)}}{t} = \lim_{t \to 0} \frac{1}{t} \cdot \frac{0 \cdot t \cdot (0^2-t^2)}{0^2+t^2} = 0$$

c) Calcule $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ y $\frac{\partial^2 f}{\partial y \partial x}(0,0)$

Sol: Usando la def. de der. direccional:

- $\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{t\to 0} \frac{\frac{\partial f}{\partial y}(t,0) - \cancel{\frac{\partial f}{\partial y}(0,0)}^{\to 0 \text{ (por (b))}}}{t}$

parte (a) $\rightarrow$ $= \lim_{t\to 0} \frac{1}{t} \cdot \frac{t^5 - 4t^3\cdot 0^2 - t\cdot 0^4}{(0^2+t^2)^2}$

$= \lim_{t\to 0} \frac{1}{\cancel{t}} \cdot \frac{t^5}{t^4} = \lim_{t\to 0} 1 = 1$

- $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{t\to 0} \frac{\frac{\partial f}{\partial x}(0,t) - \cancel{\frac{\partial f}{\partial x}(0,0)}^{\to 0 \text{ (por (b))}}}{t}$

parte (a) $\rightarrow$ $= \lim_{t\to 0} \frac{1}{\cancel{t}} \cdot \frac{\cancel{t}(0^4 + 4\cdot 0^2 t^2 - t^4)}{(0^2+t^2)^2}$

$= \lim_{t\to 0} \frac{-\cancel{t^4}}{\cancel{t^4}} = -1$

d) ¿Qué voló?

Sol: ~~Usa......~~ El teo. de Schwarz exige clase C^2.

Notemos que f no es C^2: si $(x,y) \neq (0,0)$, se tiene que $\frac{\partial^2 f}{\partial x^2}(x,y) = -\frac{4xy^3(x^2-3y^2)}{(x^2+y^2)^3}$.

Luego: $\cdot\ \frac{\partial^2 f}{\partial x^2}(0,0) = \lim_{t\to 0} \frac{\frac{\partial f}{\partial x}(t,0) - \cancel{\frac{\partial f}{\partial x}(0,0)}^{\to 0}}{t} = \lim_{t\to 0} \frac{1}{t} \cdot \frac{0\cdot(t^4 + 4t^2\cdot 0^2 - 0^4)}{(t^2+0^2)^2} = 0$

Por otro lado, si tomamos $x_n \doteq \frac{1}{n}$, $y_n \doteq \frac{1}{n}$ ($x_n \to 0 \wedge y_n \to 0$):

$\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{n},\frac{1}{n}\right) = \frac{-4\frac{1}{n^4}\left(\frac{1}{n^2} - \frac{3}{n^2}\right)}{8\cdot\frac{1}{n^6}} = 1 \underset{n\to\infty}{\not\to} 0 = \frac{\partial^2 f}{\partial x^2}(0,0)$

(i.e., $\frac{\partial^2 f}{\partial x^2}(\cdot,\cdot)$ no es continua)

73)
$$\begin{cases} x^2y + \operatorname{sen}(xyz) + z^2 = 1 \\ e^{yz} + xz = 1 \end{cases}$$

implícita

a) Sol: Primero, recordemos el ~~enunciado~~ de la f. ~~implícita~~:

Teo: (Función implícita)

Sea $f:\mathbb{R}^{m+\ell}\to\mathbb{R}^\ell$ de clase $\mathcal{C}^1$, sean $\bar{x}\in\mathbb{R}^m$, $\bar{y}\in\mathbb{R}^\ell$ tales que $f(\bar{x},\bar{y}) = \bar{C}$, con $\bar{C}\in\mathbb{R}^\ell$. Si $D_y f(\bar{x},\bar{y})$ $(\in\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell))$ es invertible, entonces $\exists\, \mathcal{U}\subseteq\mathbb{R}^m$ con $\bar{x}\in\mathcal{U}$, y $\exists\, \Phi:\mathcal{U}\to\mathbb{R}^\ell$ de clase $\mathcal{C}^1$ tal que

$$f(x,\Phi(x)) = \bar{C},\ \forall x\in\mathcal{U}.$$

Además,

$$D\Phi(x) = -\left[D_y f(x,\Phi(x))\right]^{-1}\cdot D_x f(x,\Phi(x)),\ \forall x\in\mathcal{U}.$$

Así, definimos $f:\mathbb{R}^{1+2}\to\mathbb{R}^2$ como

$$f(x,y,z) \doteq (x^2y + \operatorname{sen}(xyz) + z^2 - 1,\ e^{yz} + xz - 1),$$

y tomamos $\bar{x}\doteq 1$, $(\bar{y},\bar{z})\doteq(1,0)$, $\bar{C}\doteq(0,0)$. Ahora vayamos verificando las hipótesis:

1) $f(\bar{x},\bar{y},\bar{z}) = \bar{C}$: Directo. En efecto,

$$f(1,1,0) = (1^2\cdot 1 + \operatorname{sen}(1\cdot 1\cdot 0) + 0^2 - 1,\ e^{1\cdot 0} + 1\cdot 0 - 1)$$
$$= (0,0)$$

2) f es C^1: Clarísimo; álg. y comp. de C^1.

3) $D_{(y,z)} f(\bar{x}, \bar{y}, \bar{z})$ es invertible.

$$D_{(y,z)} f(\bar{x}, \bar{y}, \bar{z}) = \begin{pmatrix} \frac{\partial f_1}{\partial y}(\bar{x}, \bar{y}, \bar{z}) & \frac{\partial f_1}{\partial z}(\bar{x}, \bar{y}, \bar{z}) \\ \frac{\partial f_2}{\partial y}(\bar{x}, \bar{y}, \bar{z}) & \frac{\partial f_2}{\partial z}(\bar{x}, \bar{y}, \bar{z}) \end{pmatrix}$$

$$= \begin{pmatrix} \bar{x}^2 + \cos(\bar{x}\bar{y}\bar{z})\bar{x}\bar{z} & \cos(\bar{x}\cdot\bar{y}\cdot\bar{z})\bar{x}\cdot\bar{y} + 2\bar{z} \\ e^{\bar{y}\bar{z}}\cdot\bar{z} & e^{\bar{y}\cdot\bar{z}}\cdot\bar{y} + \bar{x} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\Rightarrow \det(D_{(y,z)} f(\bar{x}, \bar{y}, \bar{z})) = 1\cdot 2 - 0\cdot 1 = 2 \neq 0$$

$$\Longrightarrow D_{(y,z)} f(\bar{x}, \bar{y}, \bar{z}) \text{ es invertible.}$$

Y así hemos reunido las hip. del teo., y se concluye lo pedido. ☒

b) Sol: Acá tendremos que usar la otra concepción del teorema. Ordenémonos: La función $g(x,y,z) = x^2+y^2+z^2$ es diferenciable

$$\Longrightarrow dg(x,y,z; d) = \langle Dg(x,y,z), d\rangle, \ \forall d \text{ con } \|d\| = 1.$$

Sabemos que la dir. tgte. es $r'(t)$. En este caso:

$$r(t) = (t, y(t), z(t)) \Rightarrow r'(t) = (1, y'(t), z'(t)),$$

para $t = 1$. Es decir, buscamos $(1, y'(1), z'(1))$.

Pero, de la 2ª conc. del teo, se tiene que:

$$D\Phi(\bar{x}) = -\left[D_{(y,z)} f(\bar{x}, \Phi(\bar{x}))\right]^{-1} \cdot D_x f(\bar{x}, \Phi(\bar{x})).$$

~~Cotidiano~~ Con $\Phi(\bar{x}) = \begin{pmatrix} y(\bar{x}) \\ z(\bar{x}) \end{pmatrix}$ (el despeje).

Notemos que: (uso que $\bar{x} = 1$)

- $D_{(y,z)} f(\bar{x}, \Phi(\bar{x})) = D_{(y,z)} f(\bar{x}, \bar{y}, \bar{z})$

$$\overset{\text{lo sacamos en (a)}}{=} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\Rightarrow \left[D_{(y,z)} f(\bar{x}, \Phi(\bar{x}))\right]^{-1} = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix}$$

- $D_x f(\bar{x}, \Phi(\bar{x})) = D_x f(\bar{x}, \bar{y}, \bar{z}) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(\bar{x}, \bar{y}, \bar{z}) \\ \frac{\partial f_2}{\partial x}(\bar{x}, \bar{y}, \bar{z}) \end{pmatrix}$

$$= \begin{pmatrix} 2\bar{x}\cdot\bar{y} + \cos(\bar{x}\cdot\bar{y} - \bar{z})\,\bar{y}\bar{z} \\ \bar{z} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y'(1) \\ z'(1) \end{pmatrix} = D\Phi(1)$$

$$= -\begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$\Rightarrow y'(1) = -2, \quad z'(1) = 0.$$

Con ello:

$$\Rightarrow r'(1) = (1, y'(1), z'(1))$$
$$= (1, -2, 0).$$

Notamos que $\|r'(1)\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$. Entonces,

$$dg\left(1,1,0 ; \frac{r'(1)}{\|r'(1)\|}\right) = \left\langle Dg(1,1,0) , \frac{r'(1)}{\|r'(1)\|} \right\rangle$$

$$= \left\langle \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}^T , \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right\rangle \quad \left(\text{pues } \nabla g(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right).$$

$$= -\frac{2}{\sqrt{5}}$$