Pauta Auxiliar #7

P1] Sea $f: \mathbb{R}^2 \to \mathbb{R}$ dif. Definimos $\varphi, \psi: \mathbb{R} \to \mathbb{R}$ como

$$\varphi(t) \doteq f(t, f(t,t)) \quad ; \quad \psi(t) \doteq f(t, f(t, f(t,t))) = f(t, \varphi(t)).$$

Calcule $\psi'(t)$.

Sol: Ya que $\psi(t) = f(t, \varphi(t))$, sería conveniente 1° calcular $\varphi'(t)$. En efecto, notemos que podemos escribir φ como la composición siguiente:

- $I: \mathbb{R} \to \mathbb{R}^2$
  $t \mapsto I(t) \doteq (t,t)$
- $G: \mathbb{R}^2 \to \mathbb{R}^2$
  $(t,s) \mapsto G(t,s) \doteq (t, f(t,s))$
- $f: \mathbb{R}^2 \to \mathbb{R}$ la del enunciado:

$$\Rightarrow \varphi = f \circ G \circ I, \text{ pues}$$

$$(f \circ G \circ I)(t) = (f \circ G)(t,t) = f(G(t,t)) = f(t, f(t,t)) = \varphi(t).$$

Gráficamente:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\varphi"] \arrow[d, "I"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{R} \\
\mathbb{R}^2 \arrow[r, "G"'] & \mathbb{R}^2 \arrow[u, "f"']
\end{tikzcd}$$

$$\boxed{\varphi = f \circ G \circ I}$$

Estamos listos para usar la regla de la cadena. 1° calcularemos cada derivada, para luego juntarlas:

(*) Todo va evaluado en (t,s).

$$\underline{I}\,\Big|\quad DI(t) = \begin{pmatrix} \frac{dI_1}{dt}(t) \\ \frac{dI_2}{dt}(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{G}\,\Big|\quad DG(t,s) = \begin{pmatrix} \frac{\partial G_1}{\partial x}(t,s) & \frac{\partial G_1}{\partial y}(t,s) \\ \frac{\partial G_2}{\partial x}(t,s) & \frac{\partial G_2}{\partial y}(t,s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\partial f}{\partial x}(t,s) & \frac{\partial f}{\partial y}(t,s) \end{pmatrix}$$

$$\underline{f}\,\Big|\quad Df(t,s) = \nabla f(t,s)^T = \left(\frac{\partial f}{\partial x}(t,s) \,,\, \frac{\partial f}{\partial y}(t,s) \right).$$

Entonces, escribiendo la R.C.:

$$\begin{aligned}
\varphi'(t) = D\varphi(t) &= D(f\circ G\circ I)(t) \\
&= Df([G\circ I](t)) \cdot D(G\circ I)(t) \quad (\text{R.C.}) \\
&= Df(t, f(t,t)) \cdot D(G\circ I)(t) \quad (\text{Reemplazar } [G\circ I](t)) \\
&= Df(t, f(t,t)) \cdot DG(I(t)) \cdot DI(t) \quad (\text{R.C.}) \\
&= Df(t, f(t,t)) \cdot DG(t,t) \cdot DI(t) \quad (\text{Reemp. } I(t)) \\
&= \left(\frac{\partial f}{\partial x}(t, f(t,t)) , \frac{\partial f}{\partial y}(t, f(t,t)) \right) \begin{pmatrix} 1 & 0 \\ \frac{\partial f}{\partial x}(t,t) & \frac{\partial f}{\partial y}(t,t) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \left(\frac{\partial f}{\partial x}(t, f(t,t)) , \frac{\partial f}{\partial y}(t, f(t,t)) \right) \cdot \left(1 \,,\, \frac{\partial f}{\partial x}(t,t) + \frac{\partial f}{\partial y}(t,t) \right)^T \\
&= \frac{\partial f}{\partial x}(t, f(t,t)) + \frac{\partial f}{\partial y}(t, f(t,t)) \cdot \left[\frac{\partial f}{\partial x}(t,t) + \frac{\partial f}{\partial y}(t,t) \right]
\end{aligned}$$

Ahora démosle con Ψ. notemos que si definimos

$$H(t,s) := (t, \varphi(s)),$$

tenemos que

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\Psi"] \arrow[d, "I"'] & \mathbb{R} \\
\mathbb{R}^2 \arrow[r, "H"'] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{R}^2 \arrow[u, "f"']
\end{tikzcd} \qquad \Psi = f \circ H \circ I$$

pues

$$(f \circ H \circ I)(t) = f \circ H(t,t) = f(t, \varphi(t)) = \Psi(t).$$

Así, por R.C.:

$$\begin{aligned}
\Psi'(t) = D\Psi(t) &= D(f \circ H \circ I)(t) \\
&= Df([H \circ I](t)) \cdot D([H \circ I])(t) \\
&= Df(t, \varphi(t)) \cdot D([H \circ I])(t) \\
&= Df(t, \varphi(t)) \cdot DH(I(t)) \cdot DI(t) \\
&= \left(\frac{\partial f}{\partial x}(t, \varphi(t)), \frac{\partial f}{\partial y}(t, \varphi(t)) \right) \cdot DH(t,t) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}$$

Calculemos $DH(t,t)$:

En general,

$$DH(t,s) = \begin{pmatrix} \frac{\partial H_1}{\partial t}(t,s) & \frac{\partial H_1}{\partial s}(t,s) \\ \frac{\partial H_2}{\partial t}(t,s) & \frac{\partial H_2}{\partial s}(t,s) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \varphi'(s) \end{pmatrix}$$

$$\Rightarrow DH(t,t) = \begin{pmatrix} 1 & 0 \\ 0 & \varphi'(t) \end{pmatrix}.$$

y así, finalmente:

$$\Rightarrow \psi'(t) = \left(\frac{\partial f}{\partial x}(t,\varphi(t)), \frac{\partial f}{\partial y}(t,\varphi(t)) \right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & \varphi'(t) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \left(\frac{\partial f}{\partial x}(t,\varphi(t)), \frac{\partial f}{\partial y}(t,\varphi(t)) \right) \cdot \left(1, \varphi'(t) \right)^T$$

$$= \frac{\partial f}{\partial x}(t,\varphi(t)) + \frac{\partial f}{\partial y}(t,\varphi(t)) \cdot \varphi'(t)$$

Pauta Aux #7

P3] La idea será usar la simetría radial para convertir la ec. de Laplace en otra ec. que sólo dependa de un parámetro. Para ello, parece sensato hacer el cambio de coord. a polares.

Definimos $F:\mathbb{R}^2 \to \mathbb{R}^2$

$$(x,y) \mapsto T(x,y) = (r,\phi)$$

con $r = \sqrt{x^2+y^2}$, $\phi = \text{Arctg}\left(\frac{y}{x}\right)$

Así, si consideramos a u como una función de dos variables (r,ϕ) que no depende de ϕ, tenemos

$$\begin{tikzcd} \mathbb{R}^2 \arrow[rr, "T"] \arrow[dr, "F"'] & & \mathbb{R} \\ & \mathbb{R}^2 \arrow[ur, "u"'] & \end{tikzcd} \qquad T = u \circ F$$

Luego, gracias a la regla de la cadena[1] se tiene que

- $$\frac{\partial T}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \cancelto{0}{\frac{\partial u}{\partial \phi}}\cdot\frac{\partial \phi}{\partial x}$$ $\frac{\partial u}{\partial \phi} = 0$, pues u no depende de ϕ

$$= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

- $$\frac{\partial T}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \cancelto{0}{\frac{\partial u}{\partial \phi}}\cdot\frac{\partial \phi}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

Notemos que $\frac{\partial u}{\partial r}$ es desconocido —aún—; es lo que queremos obtener. Pero podemos calcular lo otro:

$$\cdot\ \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right) = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = \frac{r\cos\phi}{r} = \cos\phi$$

$$\cdot\ \frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\right) = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{r\,\mathrm{sen}\,\phi}{r} = \mathrm{sen}\,\phi$$

$\otimes$ Con este c. de var.,
$x = r\cos\phi$
$y = r\,\mathrm{sen}\,\phi$

$$\Rightarrow\ \frac{\partial T}{\partial x} = \frac{\partial u}{\partial r}\cdot\cos\phi\ ;\ \frac{\partial T}{\partial y} = \frac{\partial u}{\partial r}\cdot\mathrm{sen}\,\phi$$

Derivando una vez más, ~~[illegible]~~, tenemos:

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial(\cos\phi)}{\partial x}\cdot\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x\partial r}\cdot\cos\phi$$

Pero:
$$\frac{\partial(\cos\phi)}{\partial x} = -\,\mathrm{sen}\,\phi\cdot\frac{\partial\phi}{\partial x}$$
$$= -\,\mathrm{sen}\,\phi\cdot\frac{\partial\left(\mathrm{Arctg}\left(\frac{y}{x}\right)\right)}{\partial x}$$
$$= -\,\mathrm{sen}\,\phi\cdot\frac{1}{\left(\frac{y}{x}\right)^2+1}\cdot\left(-\frac{y}{x^2}\right)$$
$$= \mathrm{sen}\,\phi\,\frac{y}{y^2+x^2} = \mathrm{sen}\,\phi\,\frac{\cancel{r}\,\mathrm{sen}\,\phi}{r\cancel{r}}$$
$$= \frac{1}{r}\,\mathrm{sen}^2\,\phi$$

Por otro lado, usando cadena:

$$\frac{\partial^2 u}{\partial x \partial r} = \frac{\partial^2 u}{\partial r^2} \cdot \frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \phi \partial r} \cdot \frac{\partial \phi}{\partial x}$$

$$= \frac{\partial^2 u}{\partial r^2} \cdot \cos\phi + 0 \cdot \frac{\partial \phi}{\partial x} \quad (u \text{ no dep. de } \phi)$$

Así:

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{1}{r} \operatorname{sen}^2\phi \frac{\partial u}{\partial r} + \cos^2\phi \frac{\partial^2 u}{\partial r^2}$$

Por otro lado, muy análogamente se tiene que

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{r} \cos^2\phi \frac{\partial u}{\partial r} + \operatorname{sen}^2\phi \frac{\partial^2 u}{\partial r^2}$$

~~Eso~~ sí, allí se usó que

$$\frac{\partial (\operatorname{sen}\phi)}{\partial y} = \cos\phi \cdot \frac{\partial}{\partial y}\left(\operatorname{Arctg}\left(\frac{y}{x}\right)\right) = \cos\phi \frac{1}{\left(\frac{y}{x}\right)^2 + 1} \cdot \frac{1}{x}$$

$$= \cancel{\cos\phi} \cdot \frac{1}{\left(\frac{\cancel{r}\operatorname{sen}\phi}{\cancel{r}\cos\phi}\right)^2 + 1} \cdot \frac{1}{r\cancel{\cos\phi}}$$

$$= \frac{1}{tg^2\phi + 1} \cdot \frac{1}{r} = \frac{1}{r} \cdot \frac{1}{\sec^2\phi}$$

$$= \frac{1}{r} \cdot \cos^2\phi$$

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Finalmente, se obtiene que

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \iff \left(\frac{1}{r}\operatorname{sen}^2\phi \frac{\partial u}{\partial r} + \cos^2\phi \frac{\partial^2 u}{\partial r^2}\right) + \left(\frac{1}{r}\cos^2\phi \frac{\partial u}{\partial r} + \operatorname{sen}^2\phi \frac{\partial^2 u}{\partial r^2}\right) = 0$$

$$\iff \frac{1}{r}(\operatorname{sen}^2\phi + \cos^2\phi)\frac{\partial u}{\partial r} + (\operatorname{sen}^2\phi + \cos^2\phi)\frac{\partial^2 u}{\partial r^2} = 0$$

$$\iff \boxed{\frac{\partial^2 u}{\partial r} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} = 0}$$

Una EDO, cosa que sí sabemos resolver.

En efecto, haciendo el c. de var. $u' = w \Rightarrow u'' = w'$

$$\Rightarrow w' + \frac{1}{r}w = 0 \Rightarrow rw' + w = 0$$

$$\Rightarrow (rw)' = 0 \Rightarrow rw = A_1 \,, \; A_1 \in \mathbb{R} \text{ por determinar}$$

$$\Rightarrow \boxed{w = \frac{A_1}{r}}$$

Deshaciendo el cambio:

$$\Rightarrow u' = \frac{A_1}{r} \Rightarrow u = \int \frac{A_1}{r}\,dr + A_2 = A_1 \ln(r) + A_2$$

Pero como $T(1) = T_1$, $T(2) = T_2$:

$$\Rightarrow \left.\begin{array}{l} T_2 = A_1 \cdot \ln(2) + A_2 \\ T_1 = A_1 \cdot \cancel{\ln(1)}^{0} + A_2 \end{array}\right\} \Rightarrow \begin{array}{l} A_1 = T_1 \\ A_2 = T_2 - T_1 \ln(2) \end{array}$$

$$\Rightarrow \boxed{T(x,y) = T_1 \ln\left(\sqrt{x^2+y^2}\right) + T_2 - T_1 \ln(2)}$$

1.- Sean $f: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^m \to \mathbb{R}^k$ ~~sean~~ diferenciables. Se define $h(x_1, \ldots, x_n) = g \circ f(x_1, \ldots, x_n)$. Entonces, $\forall\, 1 \leq j \leq n$,

$$\frac{\partial h}{\partial x_j}(x_1, \ldots, x_n) = \sum_{i=1}^{m} \frac{\partial g}{\partial y_i}\left(f(x_1, \ldots, x_n)\right) \frac{\partial f_i}{\partial x_j}(x_1, \ldots, x_n)$$

P2)

Sea $f:\mathbb{R}\to\mathbb{R}$ dif. y tal que $f(0)=0$, $f'(0)=1$. Consideremos $g:\mathbb{R}^2\to\mathbb{R}$ dif., con $\nabla g(0,0)=(1,3)^T$. Se define $h:\mathbb{R}^3\to\mathbb{R}$ como

$$h(x,y,z)\doteq g\left(f(x)+f(y)^2,\ f(x)+f(y)^2+f(z)^3\right).$$

Obtenga $\nabla h(0,0,0)$.

__Sol__: Calcularemos cada componente de $\nabla h(0,0,0)$, valiéndonos del corolario de la R.C. expuesto en la pauta de la __P3)__.

Si definimos $u:\mathbb{R}^3\to\mathbb{R}$, $v:\mathbb{R}^3\to\mathbb{R}$ como

$$u(x,y,z)\doteq f(x)+f(y)^2$$
$$v(x,y,z)\doteq f(x)+f(y)^2+f(z)^3,$$

tenemos que

$$h(x,y,z)=g(u(x,y,z),v(x,y,z)).$$

Así, por el corolario,

- $$\frac{\partial h}{\partial x}(x,y,z)=\frac{\partial g}{\partial u}(u,v)\cdot\frac{\partial u}{\partial x}(x,y,z)+\frac{\partial g}{\partial v}(u,v)\cdot\frac{\partial v}{\partial x}(x,y,z)$$

$$=\frac{\partial g}{\partial x}\left(f(x)+f(y)^2,\ f(x)+f(y)^2+f(z)^3\right)\cdot f'(x)$$

$$+\ \frac{\partial g}{\partial y}\left(f(x)+f(y)^2,\ f(x)+f(y)^2+f(z)^3\right)\cdot f'(x)$$

$$\Rightarrow \frac{\partial h}{\partial x}(0,0,0)=\frac{\partial g}{\partial x}(0,0)\cdot 1+\frac{\partial g}{\partial y}(0,0)\cdot 1=1\cdot 1+3\cdot 1=4$$

$\cdot\ \dfrac{\partial h}{\partial y}(x,y,z) = \dfrac{\partial g}{\partial u}(u,v)\cdot\dfrac{\partial u}{\partial y}(x,y,z) + \dfrac{\partial g}{\partial v}(u,v)\cdot\dfrac{\partial v}{\partial y}(x,y,z)$

$\cancel{= \dfrac{\partial g}{\partial x}(\ldots)\cdot 2f'(y) + \dfrac{\partial g}{\partial y}(\ldots)}$

$= \dfrac{\partial g}{\partial x}(\ldots)\cdot 2f(y)\cdot f'(y) + \dfrac{\partial g}{\partial y}(\ldots)\cdot 2f(y)\cdot f'(y)$

$\Rightarrow\ \dfrac{\partial h}{\partial y}(0,0,0) = \dfrac{\partial g}{\partial x}(0,0)\cdot 2\cdot 0\cdot 1 + \dfrac{\partial g}{\partial y}(\ldots)\cdot 2\cdot 0\cdot 1 = 0$

$\cdot\ \dfrac{\partial h}{\partial z}(x,y,z) = \dfrac{\partial g}{\partial u}(u,v)\cdot\dfrac{\partial u}{\partial z}(x,y,z) + \dfrac{\partial g}{\partial v}(u,v)\cdot\dfrac{\partial v}{\partial z}(x,y,z)$

$\cancel{= \dfrac{\partial g}{\partial x}(\ldots)\cdot 3f(z)\cdot f'(z)}$

$= \dfrac{\partial g}{\partial x}(\ldots)\cdot 0 + \dfrac{\partial g}{\partial y}(\ldots)\cdot 3f(z)\cdot f'(z)$

$\Rightarrow\ \dfrac{\partial h}{\partial z}(0,0,0) = \dfrac{\partial g}{\partial x}(0,0)\cdot 0 + \dfrac{\partial g}{\partial y}(0,0)\cdot 3\cdot 0\cdot 1 = 0$

$$\Rightarrow \nabla h(0,0,0) = \begin{pmatrix} \frac{\partial h}{\partial x}(0,0,0) \\ \frac{\partial h}{\partial y}(0,0,0) \\ \frac{\partial h}{\partial z}(0,0,0) \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

P4 | **Teo**: (TVM en VV)

Sea $g:\mathbb{R}^d \to \mathbb{R}$ diferenciable; sean $x, y \in \mathbb{R}^d$.

Entonces, $\exists z_0 \in [x,y]$ tq $\nabla g(z_0)^T (y-x) = g(y) - g(x)$.

Dem: Definimos $h: \mathbb{R} \to \mathbb{R}^d$ como

$$h(t) \doteq x + t(y-x), \quad \forall t \in \mathbb{R}.$$

Nótese que $h'(t) = y - x$, cosa que si no me creen, se puede probar por definición.

De esta forma, si se define $f \doteq g \circ h$, se tiene:

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "f"] \arrow[dr, "h"'] & & \mathbb{R} \\ & \mathbb{R}^d \arrow[ur, "g"'] & \end{tikzcd} \qquad f = g \circ h$$

Como f es de una variable, tenemos derecho a usar TVM en $\mathbb{R}$, sobre el intervalo $[0,1]$. Así:

$$\Rightarrow \exists t_0 \in [0,1] : f'(t_0) = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0).$$

Pero:

⊛ $f(1) = g(\cancel{x} + 1 \cdot (y - \cancel{x})) = g(y)$

⊛ $f(0) = g(x + 0 \cdot (y-x)) = g(x)$

Por otro lado, usando RC:

$$\Rightarrow f'(t_0) = Df(t_0) = D(g \circ h)(t_0)$$
$$= Dg(h(t_0)) \cdot Dh(t_0)$$
$$= \nabla g(h(t_0)) \cdot (y-x)$$
$$= \nabla g(x + t_0(y-x)) \cdot (y-x)$$

$$\Rightarrow \exists t_0 \in [0,1] : \nabla g(x + t_0(y-x)) \cdot (y-x) = g(y) - g(x)$$

$$\Leftrightarrow \exists t_0 \in [0,1] : \nabla g(\underbrace{(1-t_0)x + t_0 y}_{\in [x,y]}) \cdot (y-x) = g(y) - g(x)$$

definimos $z_0 \doteq (1-t_0)x + t_0 y$

$$\Leftrightarrow \exists z_0 \in [x,y] : \nabla g(z_0) \cdot (y-x) = g(y) - g(x)$$

$\blacksquare$

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