UNIVERSITY OF LIVERPOOL

PHYS370 – Advanced Electromagnetism

Part 3: Electromagnetic Waves in Conducting Media

Electromagnetic Wave Equation

Recall that in a "simple" dielectric material, we derived the wave equations:

$$\nabla^2 \vec{E} - \mu\varepsilon \ddot{\vec{E}} = 0 \quad (1)$$
$$\nabla^2 \vec{B} - \mu\varepsilon \ddot{\vec{B}} = 0 \quad (2)$$

To derive these equations, we used Maxwell's equations with the assumptions that the charge density ρ and current density J were zero, and that the permeability μ and permittivity ε were constants.

We found that the above equations had plane-wave solutions, with phase velocity:

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \quad (3)$$

Maxwell's equations imposed additional constraints on the directions and relative amplitudes of the electric and magnetic fields.

Electromagnetic Wave Equation in Conductors

How are the wave equations (and their solutions) modified for the case of electrically conducting media?

We shall restrict our analysis to the case of ohmic conductors, which are defined by:

$$\vec{J} = \sigma \vec{E} \quad (4)$$

where σ is a constant, the conductivity of the material.

All we need to do is substitute from equation (4) into Maxwell's equations, then proceed as for the case of a dielectric...

Plane Monochromatic Wave in a Conducting Material

In our "simple" conductor, Maxwell's equations take the form:

$$\nabla \cdot \vec{E} = 0 \quad (5)$$
$$\nabla \cdot \vec{B} = 0 \quad (6)$$
$$\nabla \times \vec{E} = -\dot{\vec{B}} \quad (7)$$
$$\nabla \times \vec{B} = \mu\varepsilon \dot{\vec{E}} + \mu \vec{J} \quad (8)$$

where $\vec{J}$ is the current density. Assuming an ohmic conductor, we can write:

$$\vec{J} = \sigma \vec{E} \quad (9)$$

so equation (8) becomes:

$$\nabla \times \vec{B} = \mu\varepsilon \dot{\vec{E}} + \mu\sigma \vec{E} \quad (10)$$

Taking the curl of equation (7) and making appropriate substitutions as before, we arrive at the wave equation:

$$\nabla^2 \vec{E} - \mu\sigma \dot{\vec{E}} - \mu\varepsilon \ddot{\vec{E}} = 0 \quad (11)$$

The wave equation for the electric field in a conducting material is (11):

$$\nabla^2 \vec{E} - \mu\sigma\dot{\vec{E}} - \mu\varepsilon\ddot{\vec{E}} = 0 \qquad (12)$$

Let us try a solution of the same form as before:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \qquad (13)$$

Remember that to find the physical field, we have to take the real part. Substituting (13) into the wave equation (11) gives the dispersion relation:

$$-\vec{k}^2 - j\omega\mu\sigma + \omega^2\mu\varepsilon = 0 \qquad (14)$$

Compared to the dispersion relation for a dielectric, the new feature is the presence of an imaginary term in σ. This means the relationship between the wave vector $\vec{k}$ and the frequency ω is a little more complicated than before.

From the dispersion relation (14), we can expect the wave vector $\vec{k}$ to have real and imaginary parts. Let us write:

$$\vec{k} = \vec{\alpha} - j\vec{\beta} \qquad (15)$$

for parallel real vectors $\vec{\alpha}$ and $\vec{\beta}$.

Substituting (15) into the dispersion relation (14) and taking real and imaginary parts, we find:

$$\alpha = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}}\right]^{1/2} \qquad (16)$$

and:

$$\beta = \frac{\omega\mu\sigma}{2\alpha} \qquad (17)$$

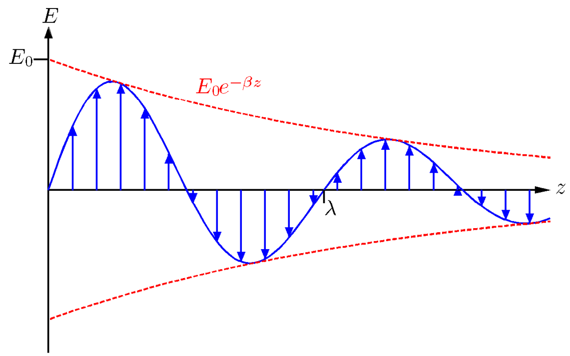
Equations (16) and (17) give the real and imaginary parts of the wave vector $\vec{k}$ in terms of the frequency ω, and the material properties μ, ε and σ.

Using equation (15) the solution (13) to the wave equation in a conducting material can be written:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{\alpha}\cdot\vec{r})} e^{-\vec{\beta}\cdot\vec{r}} \qquad (18)$$

The first exponential factor, $e^{j(\omega t - \vec{\alpha}\cdot\vec{r})}$ gives the usual plane-wave variation of the field with position $\vec{r}$ and time t; note that the conductivity of the material affects the wavelength for a given frequency.

The second exponential factor, $e^{-\vec{\beta}\cdot\vec{r}}$ gives an exponential decay in the amplitude of the wave...



In a "simple" non-conducting material there is no exponential decay of the amplitude: electromagnetic waves can travel for ever, without any loss of energy.

If the wave enters an electrical conductor, however, we can expect very different behaviour. The electrical field in the wave will cause currents to flow in the conductor. When a current flows in a conductor (assuming it is not a superconductor) there will be some energy changed into heat. This energy must come from the wave. Therefore, we expect the wave gradually to decay.

The varying electric field must have a magnetic field associated with it. Presumably, the magnetic field has the same wave vector and frequency as the electric field: this is the only way we can satisfy Maxwell's equations for all positions and times. Therefore, we try a solution of the form:

$$\vec{B}(\vec{r},t) = \vec{B}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \tag{19}$$

Now we use Maxwell's equation (7):

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{20}$$

which gives:

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0 \tag{21}$$

or:

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \tag{22}$$

The magnetic field in a wave in a conducting material is related to the electric field by (22):

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \tag{23}$$

As in a non-conducting material, the electric and magnetic fields are perpendicular to the direction of motion (the wave is a transverse wave) and are perpendicular to each other.

But there is a new feature, because the wave vector is complex.

In a non-conducting material, the electric and magnetic fields were in phase: the expressions for the fields both had the same phase angle ϕ_0. In complex notation, the complex phase angles of the field amplitudes $\vec{E}_0$ and $\vec{B}_0$ were the same.

In a conductor, the complex phase of $\vec{k}$ gives a phase difference between the electric and magnetic fields.

In a conducting material, there is a difference between the phase angles of $\vec{E}_0$ and $\vec{B}_0$, given by the phase angle ϕ of $\vec{k}$. This is:

$$\tan\phi = \frac{\beta}{\alpha} \tag{24}$$