

Solutions to Problems in Jackson,
Classical Electrodynamics, Third Edition

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Chapter 2

Problem 2.1

A point charge q is brought to a position a distance d away from an infinite plane conductor held at zero potential. Using the method of images, find:

- (a) the surface-charge density induced on the plane, and plot it;
- (b) the force between the plane and the charge by using Coulomb's law for the force between the charge and its image;
- (c) the total force acting on the plane by integrating $\sigma^2/2\epsilon_0$ over the whole plane;
- (d) the work necessary to remove the charge q from its position to infinity;
- (e) the potential energy between the charge q and its image (compare the answer to part d and discuss).
- (f) Find the answer to part d in electron volts for an electron originally one angstrom from the surface.

(a) We'll take d to be in the z direction, so the charge q is at $(x, y, z) = (0, 0, d)$. The image charge is $-q$ at $(0, 0, -d)$. The potential at a point \mathbf{r} is

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\mathbf{r} - d\mathbf{k}|} - \frac{1}{|\mathbf{r} + d\mathbf{k}|} \right]$$

The surface charge induced on the plane is found by differentiating this:

$$\begin{aligned}
\sigma &= -\epsilon_0 \frac{d\Phi}{dz} \Big|_{z=0} \\
&= -\frac{q}{4\pi} \left[\frac{-(z-d)}{|\mathbf{r}+d\mathbf{k}|^3} + \frac{(z+d)}{|\mathbf{r}+d\mathbf{k}|^3} \right] \Big|_{z=0} \\
&= -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}} \tag{1}
\end{aligned}$$

We can check this by integrating this over the entire xy plane and verifying that the total charge is just the value $-q$ of the image charge:

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x,y) dx dy &= -\frac{qd}{2\pi} \int_0^{\infty} \int_0^{2\pi} \frac{r d\psi dr}{(r^2+d^2)^{3/2}} \\
&= -qd \int_0^{\infty} \frac{r dr}{(r^2+d^2)^{3/2}} \\
&= -\frac{qd}{2} \int_{d^2}^{\infty} u^{-3/2} du \\
&= -\frac{qd}{2} \left[-2u^{-1/2} \right]_{d^2}^{\infty} \\
&= -q \quad \checkmark
\end{aligned}$$

(b) The point of this problem is that, for points above the z axis, it doesn't matter whether there is a charge $-q$ at $(0,0,d)$ or an infinite grounded sheet at $z=0$. Physics above the z axis is exactly the same whether we have the charge or the sheet. In particular, the force on the original charge is the same whether we have the charge or the sheet. That means that, if we assume the sheet is present instead of the charge, it will feel a reaction force equal to what the image charge would feel if it were present instead of the sheet. The force on the image charge would be just $F = q^2/16\pi\epsilon_0 d^2$, so this must be what the sheet feels.

(c) Total force on sheet

$$\begin{aligned}
&= \frac{1}{2\epsilon_0} \int_0^{\infty} \int_0^{2\pi} \sigma^2 dA \\
&= \frac{q^2 d^2}{4\pi\epsilon_0} \int_0^{\infty} \frac{r dr}{(r^2+d^2)^3} \\
&= \frac{q^2 d^2}{8\pi\epsilon_0} \int_{d^2}^{\infty} u^{-3} du \\
&= \frac{q^2 d^2}{8\pi\epsilon_0} \left[-\frac{1}{2} u^{-2} \right]_{d^2}^{\infty} \\
&= \frac{q^2 d^2}{8\pi\epsilon_0} \left[\frac{1}{2} d^{-4} \right]
\end{aligned}$$

$$= \frac{q^2}{16\pi\epsilon_0 d^2}$$

in accordance with the discussion and result of part b.

(d) Work required to remove charge to infinity

$$\begin{aligned} &= \frac{q^2}{4\pi\epsilon_0} \int_d^\infty \frac{dz}{(z+d)^2} \\ &= \frac{q^2}{4\pi\epsilon_0} \int_{2d}^\infty u^{-2} du \\ &= \frac{q^2}{4\pi\epsilon_0} \frac{1}{2d} \\ &= \frac{q^2}{8\pi\epsilon_0 d} \end{aligned}$$

(e) Potential energy between charge and its image

$$= \frac{q^2}{8\pi\epsilon_0 d}$$

equal to the result in part d.

(f)

$$\begin{aligned} \frac{q^2}{8\pi\epsilon_0 d} &= \frac{(1.6 \cdot 10^{-19} \text{ coulombs})^2}{8\pi(8.85 \cdot 10^{-12} \text{ coulombs V}^{-1}\text{m}^{-1})(10^{-10} \text{ m})} \\ &= 7.2 \cdot (1.6 \cdot 10^{-19} \text{ coulombs} \cdot 1 \text{ V}) \\ &= 7.2 \text{ eV.} \end{aligned}$$

Problem 2.2

Using the method of images, discuss the problem of a point charge q *inside* a hollow, grounded, conducting sphere of inner radius a . Find

- (a) the potential inside the sphere;
- (b) the induced surface-charge density;
- (c) the magnitude and direction of the force acting on q .
- (d) Is there any change in the solution if the sphere is kept at a fixed potential V ? If the sphere has a total charge Q on its inner and outer surfaces?

Problem 2.3

A straight-line charge with constant linear charge density λ is located perpendicular to the $x - y$ plane in the first quadrant at (x_0, y_0) . The intersecting planes $x = 0, y \geq 0$ and $y = 0, x \geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential, fields, and surface charges in the first quadrant.

- (a) The well-known potential for an isolated line charge at (x_0, y_0) is $\Phi(x, y) = (\lambda/4\pi\epsilon_0) \ln(R^2/r^2)$, where $r^2 = (x - x_0)^2 + (y - y_0)^2$ and R is a constant. Determine the expression for the potential of the line charge in the presence of the intersecting planes. Verify explicitly that the potential and the tangential electric field vanish on the boundary surface.
- (b) Determine the surface charge density σ on the plane $y = 0, x \geq 0$. Plot σ/λ versus x for $(x_0 = 2, y_0 = 1), (x_0 = 1, y_0 = 1)$, and $(x_0 = 1, y_0 = 2)$.
- (c) Show that the total charge (per unit length in z) on the plane $y = 0, x \geq 0$ is

$$Q_x = -\frac{2}{\pi} \lambda \tan^{-1} \left(\frac{x_0}{y_0} \right)$$

What is the total charge on the plane $x = 0$?

- (d) Show that far from the origin [$\rho \gg \rho_0$, where $\rho = \sqrt{x^2 + y^2}$ and $\rho_0 = \sqrt{x_0^2 + y_0^2}$] the leading term in the potential is

$$\Phi \rightarrow \Phi_{\text{asym}} = \frac{4\lambda}{\pi\epsilon_0} \frac{(x_0)(y_0)(xy)}{\rho^4}.$$

Interpret.

(a) The potential can be made to vanish on the specified boundary surfaces by pretending that we have three image line charges. Two image charges have charge density $-\lambda$ and exist at the locations obtained by reflecting the original image charge across the x and y axes, respectively. The third image charge has charge density $+\lambda$ and exists at the location obtained by reflecting the original charge through the origin. The resulting potential in the first quadrant is

$$\begin{aligned} \Phi(x, y) &= \frac{\lambda}{4\pi\epsilon_0} \left(\ln \frac{R^2}{r_1^2} - \ln \frac{R^2}{r_2^2} - \ln \frac{R^2}{r_3^2} + \ln \frac{R^2}{r_4^2} \right) \\ &= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_2 r_3}{r_1 r_4} \end{aligned} \quad (2)$$

where

$$r_1^2 = [(x - x_0)^2 + (y - y_0)^2] \quad r_2^2 = [(x + x_0)^2 + (y - y_0)^2]$$

$$r_3^2 = [(x - x_0)^2 + (y + y_0)^2] \quad r_4^2 = [(x + x_0)^2 + (y + y_0)^2].$$

From this you can see that

- when $x = 0$, $r_1 = r_2$ and $r_3 = r_4$
- when $y = 0$, $r_1 = r_3$ and $r_2 = r_4$

and in both cases the argument of the logarithm in (2) is unity.

(b)

$$\begin{aligned} \sigma &= -\epsilon_0 \frac{d}{dy} \Phi \\ &= -\frac{\lambda}{2\pi} \left(\frac{1}{r_2} \frac{dr_2}{dy} + \frac{1}{r_3} \frac{dr_3}{dy} - \frac{1}{r_1} \frac{dr_1}{dy} - \frac{1}{r_4} \frac{dr_4}{dy} \right) \Big|_{y=0} \end{aligned}$$

We have $dr_1/dy = (y - y_0)/r_1$ and similarly for the other derivatives, so

$$\begin{aligned} \sigma &= -\frac{\lambda}{2\pi} \left(\frac{y - y_0}{r_2^2} + \frac{y + y_0}{r_3^2} - \frac{y - y_0}{r_1^2} - \frac{y + y_0}{r_4^2} \right) \Big|_{y=0} \\ &= -\frac{y_0 \lambda}{\pi} \left(\frac{1}{(x - x_0)^2 + y_0^2} - \frac{1}{(x + x_0)^2 + y_0^2} \right) \end{aligned}$$

(c) Total charge per unit length in z

$$\begin{aligned} Q_x &= \int_0^\infty \sigma dx \\ &= -\frac{y_0 \lambda}{\pi} \left[\int_0^\infty \frac{dx}{(x - x_0)^2 + y_0^2} - \int_0^\infty \frac{dx}{(x + x_0)^2 + y_0^2} \right] \end{aligned}$$

For the first integral the appropriate substitution is $(x - x_0) = y_0 \tan u$, $dx = y_0 \sec^2 u du$. A similar substitution works in the second integral.

$$\begin{aligned} &= -\frac{\lambda}{\pi} \left[\int_{\tan^{-1} \frac{-x_0}{y_0}}^{\pi/2} du - \int_{\tan^{-1} \frac{x_0}{y_0}}^{\pi/2} du \right] \\ &= -\frac{\lambda}{\pi} \left[\frac{\pi}{2} - \tan^{-1} \frac{-x_0}{y_0} - \frac{\pi}{2} + \tan^{-1} \frac{x_0}{y_0} \right] \\ &= -\frac{2\lambda}{\pi} \tan^{-1} \frac{x_0}{y_0}. \end{aligned} \tag{3}$$

The calculations are obviously symmetric with respect to x_0 and y_0 . The total charge on the plane $x = 0$ is (3) with x_0 and y_0 interchanged:

$$Q_y = -\frac{2\lambda}{\pi} \tan^{-1} \frac{y_0}{x_0}$$

Since $\tan^{-1} x - \tan^{-1}(1/x) = \pi/2$ the total charge induced is

$$Q = -\lambda$$

which is, of course, also the sum of the charge per unit length of the three image charges.

(d) We have

$$\Phi = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{r_2^2 r_3^2}{r_1^2 r_4^2}$$

Far from the origin,

$$\begin{aligned} r_1^2 &= [(x - x_0)^2 + (y - y_0)^2] \\ &= \left[x^2 \left(1 - \frac{x_0}{x}\right)^2 + y^2 \left(1 - \frac{y_0}{y}\right)^2 \right] \\ &\approx \left[x^2 \left(1 - 2\frac{x_0}{x}\right) + y^2 \left(1 - 2\frac{y_0}{y}\right) \right] \\ &= [x^2 - 2x_0x + y^2 - 2y_0y] \\ &= (x^2 + y^2) \left[1 - 2\frac{xx_0 + yy_0}{x^2 + y^2} \right] \end{aligned}$$

Similarly,

$$\begin{aligned} r_2^2 &= (x^2 + y^2) \left[1 - 2\frac{-xx_0 + yy_0}{x^2 + y^2} \right] \\ r_3^2 &= (x^2 + y^2) \left[1 - 2\frac{xx_0 - yy_0}{x^2 + y^2} \right] \\ r_4^2 &= (x^2 + y^2) \left[1 - 2\frac{-xx_0 - yy_0}{x^2 + y^2} \right] \end{aligned}$$

Next,

$$\begin{aligned} r_1^2 r_4^2 &= (x^2 + y^2)^2 \left[1 - 4\frac{(xx_0 + yy_0)^2}{(x^2 + y^2)^2} \right] \\ r_2^2 r_3^2 &= (x^2 + y^2)^2 \left[1 - 4\frac{(xx_0 - yy_0)^2}{(x^2 + y^2)^2} \right] \end{aligned}$$

so

$$\Phi = \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{1 - 4\frac{(xx_0 - yy_0)^2}{(x^2 + y^2)^2}}{1 - 4\frac{(xx_0 + yy_0)^2}{(x^2 + y^2)^2}} \right].$$

The $(x^2 + y^2)$ term in the denominator grows much more quickly than the $(xx_0 + yy_0)$ term, so in the asymptotic limit we can use $\ln(1 + \epsilon) \approx \epsilon$ to find

$$\begin{aligned} \Phi &= \frac{\lambda}{4\pi\epsilon_0} \left[-4\frac{(xx_0 - yy_0)^2}{(x^2 + y^2)^2} + 4\frac{(xx_0 + yy_0)^2}{(x^2 + y^2)^2} \right] \\ &= \frac{\lambda}{4\pi\epsilon_0} \left[\frac{-4(x^2x_0^2 + y^2y_0^2 - 2xyx_0y_0) + 4(x^2x_0^2 + y^2y_0^2 + 2xyx_0y_0)}{(x^2 + y^2)^2} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{\lambda}{4\pi\epsilon_0} \left[\frac{16xyx_0y_0}{(x^2 + y^2)^2} \right] \\
&= \frac{4\lambda}{\pi\epsilon_0} \frac{(xy)(x_0y_0)}{(x^2 + y^2)^2}. \quad \checkmark
\end{aligned}$$

Problem 2.4

A point charge is placed a distance $d > R$ from the center of an equally charged, isolated, conducting sphere of radius R .

- (a) Inside of what distance from the surface of the sphere is the point charge attracted rather than repelled by the charged sphere?
- (b) What is the limiting value of the force of attraction when the point charge is located a distance $a (= d - R)$ from the surface of the sphere, if $a \ll R$?
- (c) What are the results for parts a and b if the charge on the sphere is twice (half) as large as the point charge, but still the same sign?

Let's call the point charge q . The charged, isolated sphere may be replaced by two image charges. One image charge, of charge $q_1 = -(R/d)q$ at radius $r_1 = R^2/d$, is needed to make the potential equal at all points on the sphere. The second image charge, of charge $q_2 = q - q_1$ at the center of the sphere, is necessary to recreate the effect of the additional charge on the sphere (the "additional" charge is the extra charge on the sphere left over after you subtract the surface charge density induced by the point charge q).

The force on the point charge is the sum of the forces from the two image charges:

$$F = \frac{1}{4\pi\epsilon_0} \left[\frac{qq_1}{\left[d - \frac{R^2}{d}\right]^2} + \frac{qq_2}{d^2} \right] \quad (4)$$

$$= \frac{q^2}{4\pi\epsilon_0} \left[\frac{-dR}{[d^2 - R^2]^2} + \frac{d^2 + dR}{d^4} \right] \quad (5)$$

As $d \rightarrow R$ the denominator of the first term vanishes, so that term wins, and the overall force is attractive. As $d \rightarrow \infty$, the denominator of both terms looks like d^4 , so the dR terms in the numerator cancel and the overall force is repulsive.

- (a) The crossover distance is found by equating the two bracketed terms in (5):

$$\begin{aligned}
\frac{dR}{[d^2 - R^2]^2} &= \frac{d^2 + dR}{d^4} \\
d^4 R &= (d + R)[d^2 - R^2]^2 \\
0 &= d^5 - 2d^3 R^2 - 2d^2 R^3 + dR^4 + R^5
\end{aligned}$$

I used GnuPlot to solve this one graphically. The root is $d/R=1.6178$.

(b) The idea here is to set $d = R + a = R(1 + a/R)$ and find the limit of (4) as $a \rightarrow 0$.

$$\begin{aligned}
F &= \frac{q^2}{4\pi\epsilon_0} \left[\frac{-R^2(1 + \frac{a}{R})}{[R^2(1 + \frac{a}{R})^2 - R^2]^2} + \frac{R^2 [(1 + \frac{a}{R})^2 + (1 + \frac{a}{R})]}{R^4(1 + \frac{a}{R})^4} \right] \\
&\approx \frac{q^2}{4\pi\epsilon_0} \left[\frac{-R^2 - aR}{4a^2 R^2} + \frac{(2R + 3a)(R - 4a)}{R^4} \right]
\end{aligned}$$

The second term in brackets approaches the constant $2/R^2$ as $a \rightarrow 0$. The first term becomes $-1/4a^2$. So we have

$$F \rightarrow -\frac{q^2}{16\pi\epsilon_0 a^2}.$$

Note that only the first image charge (the one required to make the sphere an equipotential) contributes to the force as $d \rightarrow a$. The second image charge, the one which represents the difference between the actual charge on the sphere and the charge induced by the first image, makes no contribution in this limit. That means that the limiting value of the force will be as above regardless of the charge on the sphere.

(c) If the charge on the sphere is twice the point charge, then $q_2 = 2q - q_1 = q(2 + R/d)$. Then (5) becomes

$$F = \frac{q^2}{4\pi\epsilon_0} \left[-\frac{dR}{[d^2 - R^2]^2} + \frac{2d^2 + dR}{d^4} \right]$$

and the relevant equation becomes

$$0 = 2d^5 - 4d^3 R^2 - 2d^2 R^3 + 2dR^4 + R^5.$$

Again I solved graphically to find $d/R = 1.43$. If the charge on the sphere is half the point charge, then

$$F = \frac{q^2}{4\pi\epsilon_0} \left[-\frac{dR}{[d^2 - R^2]^2} + \frac{d^2 + 2dR}{2d^4} \right]$$

and the equation is

$$0 = d^5 - 2d^3 R^2 - 4d^2 R^3 + dR^4 + 2R^5.$$

The root of this one is $d/R=1.88$.

Problem 2.5

- (a) Show that the work done to remove the charge q from a distance $r > a$ to infinity against the force, Eq. (2.6), of a grounded conducting sphere is

$$W = \frac{q^2 a}{8\pi\epsilon_0(r^2 - a^2)}.$$

Relate this result to the electrostatic potential, Eq. (2.3), and the energy discussion of Section 1.11.

- (b) Repeat the calculation of the work done to remove the charge q against the force, Eq. (2.9), of an isolated charged conducting sphere. Show that the work done is

$$W = \frac{1}{4\pi\epsilon_0} \left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r} \right].$$

Relate the work to the electrostatic potential, Eq. (2.8), and the energy discussion of Section 1.11.

- (a) The force is

$$|F| = \frac{q^2 a}{4\pi\epsilon_0} \frac{1}{y^3(1 - a^2/y^2)^2}$$

directed radially inward. The work is

$$\begin{aligned} W &= - \int_r^\infty F dy & (6) \\ &= \frac{q^2 a}{4\pi\epsilon_0} \int_r^\infty \frac{dy}{y^3(1 - a^2/y^2)^2} \\ &= \frac{q^2 a}{4\pi\epsilon_0} \int_r^\infty \frac{y dy}{(y^2 - a^2)^2} \\ &= \frac{q^2 a}{4\pi\epsilon_0} \int_{r^2 - a^2}^\infty \frac{du}{2u^2} \\ &= \frac{q^2 a}{4\pi\epsilon_0} \left[-\frac{1}{2u} \right]_{r^2 - a^2}^\infty \\ &= \frac{q^2 a}{8\pi\epsilon_0(r^2 - a^2)} & (7) \end{aligned}$$

To relate this to earlier results, note that the image charge $q' = -(a/r)q$ is located at radius $r' = a^2/r$. The potential energy between the point charge and

its image is

$$\begin{aligned}
 PE &= \frac{1}{4\pi\epsilon_0} \left(\frac{qq'}{|r - r'|} \right) \\
 &= \frac{1}{4\pi\epsilon_0} \left(\frac{-q^2 a}{r(r - a^2/r)} \right) \\
 &= \frac{1}{4\pi\epsilon_0} \left(\frac{-q^2 a}{r^2 - a^2} \right)
 \end{aligned} \tag{8}$$

Result (7) is only half of (8). This would seem to violate energy conservation. It would seem that we could start with the point charge at infinity and allow it to fall in to a distance r from the sphere, liberating a quantity of energy (8), which we could store in a battery or something. Then we could expend an energy equal to (7) to remove the charge back to infinity, at which point we would be back where we started, but we would still have half of the energy saved in the battery. It would seem that we could keep doing this over and over again, storing up as much energy in the battery as we pleased.

I think the problem is with equation (8). The traditional expression $q_1 q_2 / 4\pi\epsilon_0 r$ for the potential energy of two charges comes from calculating the work needed to bring one charge from infinity to a distance r from the other charge, and it is assumed that the other charge does not move and keeps a constant charge during the process. But in this case one of the charges is a fictitious image charge, and as the point charge q is brought in from infinity the image charge moves out from the center of the sphere, and its charge increases. So the simple expression doesn't work to calculate the potential energy of the configuration, and we should take (7) to be the correct result.

(b) In this case there are two image charges: one of the same charge and location as in part a, and another of charge $Q - q'$ at the origin. The work needed to remove the point charge q to infinity is the work needed to remove the point charge from its image charge, plus the work needed to remove the point charge from the extra charge at the origin. We calculated the first contribution above. The second contribution is

$$\begin{aligned}
 - \int_r^\infty \frac{q(Q - q')dy}{4\pi\epsilon_0 y^2} &= - \frac{1}{4\pi\epsilon_0} \int_r^\infty \left[\frac{qQ}{y^2} + \frac{q^2 a}{y^3} \right] dy \\
 &= - \frac{1}{4\pi\epsilon_0} \left[-\frac{qQ}{y} - \frac{q^2 a}{2y^2} \right]_r^\infty \\
 &= - \frac{1}{4\pi\epsilon_0} \left[\frac{qQ}{r} + \frac{q^2 a}{2r^2} \right]
 \end{aligned}$$

so the total work done is

$$W = \frac{1}{4\pi\epsilon_0} \left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r} \right].$$

Review of Green's Functions

Some problems in this and other chapters use the Green's function technique. It's useful to review this technique, and also to establish my conventions since I define the Green's function a little differently than Jackson.

The whole technique is based on the divergence theorem. Suppose $\mathbf{A}(\mathbf{x})$ is a vector valued function defined at each point \mathbf{x} within a volume V . Then

$$\int_V (\nabla \cdot \mathbf{A}(\mathbf{x}')) dV' = \oint_S \mathbf{A}(\mathbf{x}') \cdot d\mathbf{A}' \quad (9)$$

where S is the (closed) surface bounding the volume V . If we take $\mathbf{A}(\mathbf{x}) = \phi(\mathbf{x})\nabla\psi(\mathbf{x})$ where ϕ and ψ are scalar functions, (9) becomes

$$\int_V [(\nabla\phi(\mathbf{x}')) \cdot (\nabla\psi(\mathbf{x}')) + \phi(\mathbf{x}')\nabla^2\psi(\mathbf{x}')] dV' = \oint_S \phi(\mathbf{x}') \left. \frac{\partial\psi}{\partial n} \right|_{\mathbf{x}'} dA'$$

where $\partial\psi/\partial n$ is the dot product of $\vec{\nabla}\psi$ with the outward normal to the surface area element. If we write down this equation with ϕ and ψ switched and subtract the two, we come up with

$$\int_V [\phi\nabla^2\psi - \psi\nabla^2\phi] dV' = \oint_S \left[\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n} \right] dA'. \quad (10)$$

This statement doesn't appear to be very useful, since it seems to require that we know ϕ over the whole volume to compute the left side, and both ϕ and $\partial\phi/\partial n$ on the boundary to compute the right side. However, suppose we could choose $\psi(\mathbf{x})$ in a clever way such that $\nabla^2\psi = \delta(\mathbf{x} - \mathbf{x}_0)$ for some point x_0 within the volume. (Since this ψ is a function of \mathbf{x} which also depends on \mathbf{x}_0 as a parameter, we might write it as $\psi_{\mathbf{x}_0}(\mathbf{x})$.) Then we could use the sifting property of the delta function to find

$$\phi(\mathbf{x}_0) = \int_V [\psi_{\mathbf{x}_0}(\mathbf{x}')\nabla^2\phi(\mathbf{x}')] dV' + \oint_S \left[\phi(\mathbf{x}') \left. \frac{\partial\psi_{\mathbf{x}_0}}{\partial n} \right|_{\mathbf{x}'} - \psi_{\mathbf{x}_0}(\mathbf{x}') \left. \frac{\partial\phi}{\partial n} \right|_{\mathbf{x}'} \right] dA'.$$

If ϕ is the scalar potential of electrostatics, we know that $\nabla^2\psi(\mathbf{x}') = -\rho(\mathbf{x}')/\epsilon_0$, so we have

$$\phi(\mathbf{x}_0) = -\frac{1}{\epsilon_0} \int_V \psi_{\mathbf{x}_0}(\mathbf{x}')\rho(\mathbf{x}')dV' + \oint_S \left[\phi(\mathbf{x}') \left. \frac{\partial\psi_{\mathbf{x}_0}}{\partial n} \right|_{\mathbf{x}'} - \psi_{\mathbf{x}_0}(\mathbf{x}') \left. \frac{\partial\phi}{\partial n} \right|_{\mathbf{x}'} \right] dA'. \quad (11)$$

Equation (11) allows us to find the potential at an arbitrary point \mathbf{x}_0 as long as we know ρ within the volume and both ϕ and $\partial\phi/\partial n$ on the boundary. Usually we do know ρ within the volume, but we only know either ϕ or $\partial\phi/\partial n$ on the boundary. This lack of knowledge can be accommodated by choosing ψ such that either its value or its normal derivative vanishes on the boundary surface, so that the term which we can't evaluate drops out of the surface integral. More specifically,

- if we know ϕ but not $\partial\phi/\partial n$ on the boundary (“Dirichlet” boundary conditions), we choose ψ such that $\psi = 0$ on the boundary. Then

$$\phi(\mathbf{x}_0) = -\frac{1}{\epsilon_0} \int_V \psi_{\mathbf{x}_0}(\mathbf{x}') \rho(\mathbf{x}') dV' + \oint_S \phi(\mathbf{x}') \left. \frac{\partial \psi_{\mathbf{x}_0}}{\partial n} \right|_{\mathbf{x}'} dA'. \quad (12)$$

- if we know $\partial\phi/\partial n$ but not ϕ on the boundary (“Neumann” boundary conditions), we choose ψ such that $\partial\psi/\partial n = 0$ on the boundary. Then

$$\phi(\mathbf{x}_0) = -\frac{1}{\epsilon_0} \int_V \psi_{\mathbf{x}_0}(\mathbf{x}') \rho(\mathbf{x}') dV' + \oint_S \phi_{\mathbf{x}_0}(\mathbf{x}') \left. \frac{\partial \phi}{\partial n} \right|_{\mathbf{x}'} dA'. \quad (13)$$

Again, in both cases the function $\psi_{\mathbf{x}_0}(\mathbf{x})$ has the property that

$$\nabla^2 \psi_{\mathbf{x}_0}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0).$$