Resumen

$$M = \sum_i m_i \quad ; \quad \vec{R}_G = \frac{1}{M}\sum_i m_i \vec{r}_i \quad ; \quad \vec{V}_G = \frac{1}{M}\sum_i m_i \vec{v}_i$$

$$M\frac{dV_G}{dt} = \sum_i \vec{F}_i \rightarrow \text{externas.}$$

$$\vec{l}_O = \vec{l}_O^{\,G} + \vec{l}_G \quad ; \quad \dot{\vec{l}}_G = \vec{\tau}_G \quad ; \quad \dot{\vec{l}}_O^{\,G} = \vec{\tau}_O^{\,G}$$

- Matriz de inercia:

$$I_{ij} = \sum_a m_a(r_a^2 \delta_{ij} - r_{a_i} r_{a_j}) \quad ; \quad I^G = \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$$

$$dm = \lambda dl = \lambda dx = \lambda dy = \lambda dz = \lambda r d\phi = \lambda dr = \dots$$

$$dm = \sigma dS = \sigma dxdy = \sigma dxdz = \sigma dydz = \sigma r dr d\phi = \dots$$

$$dm = \rho dV = \rho dxdydz = \rho r dr d\phi dz = \rho r^2 \sin\theta \, dr \, d\theta \, d\phi$$

- Rotación en torno a G:

$$M\vec{R}_G \times \ddot{\vec{R}}_G + I^G \dot{\vec{\Omega}} = \vec{\tau}_O$$

- Rotación en torno a un punto fijo P:

$$\vec{l}_P = I^P \vec{\Omega} \Rightarrow \vec{\tau}_P = I^P \dot{\vec{\Omega}}$$

- Steiner:

$$I^P = \sum_a m_a \begin{pmatrix} y_a^2+z_a^2 & -x_a y_a & -x_a z_a \\ -x_a y_a & x_a^2+z_a^2 & -y_a z_a \\ -x_a z_a & -y_a z_a & x_a^2+y_a^2 \end{pmatrix} + M \begin{pmatrix} x^2+y^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$$

- Energía:

↳ G: $K = \frac{1}{2} M v_G^2 + \frac{1}{2} \vec{\Omega} \cdot \mathbb{I}^G \cdot \vec{\Omega}$

↳ Punto fijo P: $K = \frac{1}{2} \vec{\Omega} \cdot \mathbb{I}^P \cdot \vec{\Omega}$

- Integrales útiles:

↳ $\int_0^{2\pi} d\phi \, \sin\phi \cos\phi = \int_0^{2\pi} d\phi \, \sin\phi = \int_0^{\pi} d\phi \, \cos\phi = 0$

↳ $\int_0^{2\pi} d\phi \, \sin^2\phi = \int_0^{2\pi} d\phi \, \cos^2\phi = \pi$

↳ Sea $f(x)$ par: $(f(-x) = f(x))$

$$\int_{-L}^{L} dx \, f(x) = 2 \int_0^{L} dx \, f(x)$$

↳ Sea $f(x)$ impar $(f(-x) = -f(x))$

$$\int_{-L}^{L} dx \, f(x) = 0$$

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(a)

$$I^G = \int dm \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} \qquad \begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

$$\Rightarrow I^G = \int_0^a dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\, (\rho r^2 \sin\theta)\, r^2 \begin{bmatrix} \sin^2\theta\sin^2\phi + \cos^2\theta & -\sin^2\theta\sin\phi\cos\phi & -\sin\theta\cos\theta\cos\phi \\ -\sin^2\theta\sin\phi\cos\phi & \sin^2\theta\cos^2\phi + \cos^2\theta & -\sin\theta\cos\theta\sin\phi \\ -\sin\theta\cos\theta\cos\phi & -\sin\theta\cos\theta\sin\phi & \sin^2\theta \end{bmatrix}$$

- Se tiene que:

$$\int_0^{2\pi} d\phi\, \sin\phi\cos\phi = \int_0^{2\pi} d\phi\, \sin\phi = \int_0^{2\pi} d\phi\, \cos\phi = 0$$

$$\int_0^{2\pi} d\phi\, \sin^2\phi = \int_0^{2\pi} d\phi\, \cos^2\phi = \pi$$

$$\Rightarrow I^G = \frac{\rho a^5}{5} \int_0^\pi d\theta \begin{bmatrix} \pi\sin^3\theta + 2\pi\cos^2\theta\sin\theta & 0 & 0 \\ 0 & \pi\sin^3\theta + 2\pi\cos^2\theta\sin\theta & 0 \\ 0 & 0 & 2\pi\sin^3\theta \end{bmatrix}$$

- Se tiene que:

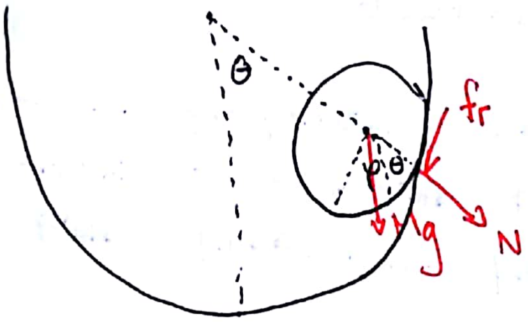
$$\int_0^\pi \sin^3\theta\, d\theta = \int_0^\pi \sin\theta(1-\cos^2\theta)\, d\theta = \frac{\cos^3\theta}{3} - \cos\theta \Big|_0^\pi = \frac{4}{3} \quad ; \quad \int_0^\pi \sin\theta\, d\theta = 2$$

$$\Rightarrow I^G = \frac{\rho a^5}{5} \int_0^\pi d\theta \begin{bmatrix} 2\pi\sin\theta - \pi\sin^3\theta & 0 & 0 \\ 0 & 2\pi\sin\theta - \pi\sin^3\theta & 0 \\ 0 & 0 & 2\pi\sin^3\theta \end{bmatrix}$$

$$\Rightarrow I^G = \frac{\rho a^5}{5} \cdot \pi \begin{bmatrix} 4 - \frac{4}{3} & 0 & 0 \\ 0 & 4 - 4/3 & 0 \\ 0 & 0 & 8/3 \end{bmatrix} = \frac{M}{\cancel{\frac{4}{3}}\cancel{\pi a^3}} \frac{\cancel{\pi} a^{\cancel{5}\,2}}{5} \cdot \frac{\cancel{8}\,2}{\cancel{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \boxed{I^G = \frac{2Ma^2}{5}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}$$

(b)



→ ∡ que gira la esfera.

$$\underbrace{b\theta}_{\text{arco que corre el CH}} = \underbrace{a(\theta+\varphi)}_{\text{arco que corre el círculo}}$$

$$\Rightarrow \varphi = \frac{(b-a)\theta}{a}$$

$$\Sigma \tau^G: \quad 0 \times m\vec{g} + a\hat{\rho} \times f_r\hat{\theta} = I^G\ddot{\varphi} = \frac{2Ma^2}{5}\cdot\frac{(b-a)}{a}\ddot{\theta}\,\hat{z}$$

$$\Rightarrow f_r = \frac{2M(b-a)}{5}\ddot{\theta}$$

$$\Sigma \vec{F}: \quad M(b-a)\ddot{\theta} = -f_r - mg\sin\theta$$

$$\Rightarrow M(b-a)\ddot{\theta} = \frac{2M(b-a)}{5}\ddot{\theta} - mg\sin\theta$$

$$\Rightarrow \frac{3M(b-a)}{5}\ddot{\theta} = -mg\sin\theta$$ | Pequeñas oscilaciones

$$\Rightarrow \frac{3\cancel{M}(b-a)}{5}\ddot{\theta} = -\cancel{M}g\theta$$

$$\Rightarrow \ddot{\theta} = -\frac{5g}{3(b-a)}\theta \Rightarrow \boxed{\omega^2 = \frac{5g}{3(b-a)}}$$

c)

- Por conservación de la energía:

↳ En $\theta =$: $E = 0$

↳ En $\theta = 0$: $E = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I \Omega^2 - Mg(b-a)$

$$\Rightarrow \frac{\cancel{M}}{2}(b-a)^2\dot{\theta}^2 + \frac{1}{\cancel{2}}\frac{\cancel{2}\cancel{M}a^2}{5}\cdot\dot{\varphi}^2 = \cancel{M}g(b-a) \quad / \quad \dot{\varphi} = \frac{(b-a)}{a}\dot{\theta}$$

$$\Rightarrow \frac{(b-a)^{\cancel{2}}\dot{\theta}^2}{2} + \frac{(b-a)^{\cancel{2}}}{5}\dot{\theta}^2 = g\cancel{(b-a)}$$

$$\Rightarrow \boxed{\dot{\theta}^2 = \frac{10g}{7(b-a)}} \quad \Rightarrow \boxed{\dot{\varphi}^2 = \frac{10g(b-a)}{7a^2}}$$

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Auxiliar 17

(a)

$$I^{o} = \underbrace{\int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}}_{I^G} + \underbrace{M \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & z^2+x^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}}_{\text{Steiner}}$$

$$I^G = \rho \int_0^R dr \int_0^{2\pi} d\phi\, r^3 \begin{vmatrix} \sin^2\phi & -\sin\phi\cos\phi & 0 \\ -\sin\phi\cos\phi & \cos^2\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow I^G = \frac{\rho R^4}{4} \begin{bmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 2\pi \end{bmatrix} = \frac{M}{\cancel{\pi R^2}} \cdot \frac{\cancel{\pi} R^{\cancel{4}\,2}}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow I^G = \frac{MR^2}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- Steiner:

$$I^{o} = \frac{MR^2}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \frac{MR^2}{4} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{I = \frac{MR^2}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

(b)

$$\sum \tau^{o}: \frac{3MR^2}{4} \cancelto{0}{\ddot{\theta}} = \frac{RMg}{2} - \frac{3RT}{2} \Rightarrow \boxed{T = \frac{Mg}{3}}$$

(c)

- Antes de que se corte la cuerda:

$\sum F$:

$$N_1 - Mg + T = 0 \Rightarrow N_1 = (T - Mg)\cdot(-1)$$

$$\Rightarrow N_1 = -\frac{Mg}{3} + Mg = +\frac{2Mg}{3}$$

- Después de que se corte la cuerda:

$$\sum F: N_2 - Mg = \frac{MR}{2}\ddot{\theta} \quad ; \quad \sum \tau: \frac{\cancel{R}}{2}\cancel{M}g = \frac{3\cancel{M}R^{\cancel{2}}}{4}\ddot{\theta}$$

$$\Rightarrow N_2 - Mg = \frac{MR}{2}\ddot{\theta} \ ; \quad \ddot{\theta} = \frac{2g}{3R}$$

$$\Rightarrow N_2 = Mg + \frac{M\cancel{R}}{\cancel{2}}\cdot\frac{\cancel{2}g}{3\cancel{R}} = \frac{4Mg}{3}$$

$$\Rightarrow \Delta N = N_2 - N_1 = \frac{2\cancel{3}Mg}{3} = \boxed{\frac{2Mg}{3}}$$

(d)

- Conservación de la energía:

$$\frac{\cancel{1}}{\cancel{2}}\frac{3\cancel{M}R^{\cancel{2}}}{4}\dot{\theta}^2 - \cancel{M}g\frac{\cancel{R}}{\cancel{2}}\theta = 0 \Rightarrow \boxed{\dot{\omega}^2 = \frac{4g}{3R}}$$

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